

Q -MANIFOLDS AND MACKENZIE THEORY

THEODORE TH. VORONOV

ABSTRACT. Double Lie algebroids were discovered by Kirill Mackenzie from the study of double Lie groupoids and were defined in terms of rather complicated conditions making use of duality theory for Lie algebroids and double vector bundles. In this paper we establish a simple alternative characterization of double Lie algebroids in a supermanifold language. Namely, we show that a double Lie algebroid in Mackenzie's sense is equivalent to a double vector bundle endowed with a pair of commuting homological vector fields of appropriate weights. Our approach helps to simplify and elucidate Mackenzie's original definition; we show how it fits into a bigger picture of equivalent structures on 'neighbor' double vector bundles. It also opens ways for extending the theory to multiple Lie algebroids, which we introduce here.

INTRODUCTION

Double Lie algebroids arose in the works on double Lie groupoids [7], [8] and in connection with an analog for Lie bialgebroids of the classical Drinfeld double of Lie bialgebras [9], [10].

K. Mackenzie put forward the idea that (the analog of) the *Drinfeld double* for a Lie bialgebroid should be a *double Lie algebroid* in the Ehresmann sense understood properly.

Recall that a 'double object' in Ehresmann's sense in a category \mathcal{C} is an object of the same type \mathcal{C} in the category \mathcal{C} . For example, a group object in the category of groups (which is, as one can see, just an Abelian group). This can be defined for certain categories where objects have structures described by diagrams. (As the above example shows, for some categories this may be defined, but be not very interesting.) 'Double Lie groupoids' are therefore groupoid objects in the category of Lie groupoids. For groupoids, differently from groups, this leads to a richer structure, not to a degeneration.

Unlike groupoids, Lie algebroids are not defined diagrammatically. Because of that, defining double objects for them could not be done directly. 'Double Lie algebroids' first appeared as the tangent objects for double Lie groupoids [7], [8]. Properties of these tangent objects were axiomatized later to give the abstract notion. The definition so obtained [9] is quite complicated. Even stating the conditions appearing there requires non-trivial applications of the duality theory for double vector bundles and of the relations between Lie algebroids and Poisson structures.

Mackenzie justified his definition of double Lie algebroids by proving the thesis quoted above, i.e., by showing that this definition is satisfied by a Lie bialgebroid generalization of Drinfeld's classical double [9], [10]. Hence there was absolutely no doubt that in [9] a correct notion was discovered. However the complexity of the original definition has somewhat hindered its further applications.

The aim of this paper is to give a simple alternative characterization of Mackenzie's double Lie algebroids. This is achieved in the language of Q -manifolds, i.e., supermanifolds endowed with an odd vector field of square zero.

An obvious part of a definition of a double Lie algebroid is, of course, two structures of ordinary Lie algebroids¹. The problem was to find compatibility conditions that have to be satisfied.

In this paper we *analyze Mackenzie's original compatibility conditions and prove that they are equivalent to the commutativity of two homological vector fields of suitable weights on a supermanifold naturally associated with a given double vector bundle*. This radically simplifies the theory and allows to extend it immediately to the multiple case, i.e., allows to introduce n -fold Lie algebroids, which before was inaccessible.

The original approach to double Lie algebroids is not made redundant by our work. In the course of the proof of our main statement, we show that Mackenzie's 'Condition III' (see Section 1 below), which pertains to a certain Lie bialgebroid, actually subsumes his other conditions. With this simplification, we show further that Mackenzie's original definition and the construction given in this paper become parts of what we call a **big picture**. It is as follows.

Recall that, say, a Lie algebra \mathfrak{g} has equivalent manifestations as a linear Poisson bracket on the coalgebra \mathfrak{g}^* , as a linear Schouten bracket on the anticoalgebra $\Pi\mathfrak{g}^*$ and as a quadratic homological vector field on the antialgebra $\Pi\mathfrak{g}$. See, for example, [24]. The vector spaces \mathfrak{g}^* , $\Pi\mathfrak{g}$ and $\Pi\mathfrak{g}^*$ are the *neighbors* of the vector space \mathfrak{g} .

To apply this idea to double Lie algebroids, consider a given double vector bundle and take all its neighbors, that is, the double vector bundles obtained by dualizations and parity reversions. There are twelve of them, including the original bundle. Assume the existence of two ordinary Lie algebroid structures on the original double vector bundle, with some natural linearity conditions. Then each of the neighbors acquires a pair of structures such as, e.g., a Lie algebroid structure and a vector field, a bracket and a vector field, etc. We can say that the structure of a 'double Lie algebroid' is manifested in various ways on all of these neighbor double vector bundles. The crucial thing is to formulate the compatibility constraint. To this end, we notice that there are exactly five cases where the induced structures are defined on the total space as opposed to the space of sections of a vector bundle. In each such case a compatibility condition comes about naturally (e.g., that the vector field should be a derivation of the Poisson bracket). All these natural conditions can be shown to be equivalent—by a certain functorial argument. It turns out that in exactly four of these cases the 'natural' compatibility condition is a reformulation of Mackenzie's Condition III, and in the remaining case it is precisely our commutativity condition.

The importance of double Lie groupoids and Lie bialgebroids, and notions related to them, such as double Lie algebroids, follows, in particular, from their natural links with Poisson geometry. The infinitesimal object for a Poisson groupoid is a Lie bialgebroid as it was shown by Mackenzie and Xu [18]. On the other hand, if one takes the cotangent bundle T^*G of a Poisson–Lie group (or a Poisson groupoid) G , it is an 'LA-groupoid', a notion intermediate between double Lie groupoids and double Lie algebroids [7]. (Recall that Poisson groupoids incorporate both Poisson–Lie groups and symplectic groupoids [27], [28].) The richness of these notions can be also seen in numerous non-obvious structures, isomorphisms and dualities arising for them, e.g., a duality theory for double and triple vector bundles, producing unexpected discrete symmetry groups [14]. We believe that methods developed in this paper will be particularly useful for all these applications.

¹In fact, four Lie algebroid structures, on the four sides of a double vector bundle; but they can be reduced to the two 'main' ones as we explain in Section 4.

We wish to stress that in our work, supermanifolds provide powerful tools that we apply to ordinary (“purely even”) objects. Although we show that everything works also in a ‘superized’ context, this was not the main goal.

The paper is organized as follows.

In Section 1 we recall the definition of double Lie algebroids.

In Section 2 we recall the description of (ordinary) Lie algebroids in the language of homological vector fields, and revise double vector bundles. In particular, we introduce partial reversions of parity.

In Section 3 we define double Lie antialgebroids and give our main statement (Theorem 1).

In Section 4 we analyze the three conditions appearing in the definition of double Lie algebroids and give a proof of Theorem 1.

In Section 5 we show how the equivalence of Mackenzie’s notion of double Lie algebroids and our notion of double Lie antialgebroids is a part of a bigger picture. *Modulo* some facts established in Section 4, this provides an alternative proof of Theorem 1.

In Section 6 we show the equivalence of Mackenzie’s and Roytenberg’s doubles of Lie bialgebroids and discuss an extension of the whole theory to the multiple case.

Terminology and notation. We use the standard language of supermanifolds. The letter Π denotes the parity reversion functor, and notation such as Φ^Π is used for linear maps induced on the *opposite* (parity reversed) objects. Commutators and similar notions are always understood in the \mathbb{Z}_2 -graded sense. A tilde over an object is used to denote its parity. A *Q-manifold* means a supermanifold endowed with a homological vector field; likewise, *P-* and *S-manifolds* mean those with a Poisson or Schouten (= odd Poisson) bracket. A *QS-manifold* (resp., a *QP-manifold*) means one with *Q-* and *S-* structures (resp., with *Q-* and *P-* structures) that are compatible (the vector field is a derivation of the bracket, cf. [4]). In general, notation and terminology are close to our paper [24]. The space of smooth sections of a vector bundle $E \rightarrow M$ is denoted by $C^\infty(M, E)$ and the space of vector fields on a manifold M , by $\text{Vect}(M)$. We wish to draw the reader’s special attention to our normally dropping the prefix ‘super-’ when this cannot cause confusion and speaking, as a rule, of ‘manifolds’ meaning supermanifolds, ‘Lie algebras’ meaning superalgebras, etc.

1. DOUBLE LIE ALGEBROIDS

Double Lie algebroids were introduced by Mackenzie in [9], see also [10], as the infinitesimal counterparts of double Lie groupoids. The latter notion is a double object in the sense of Ehresmann, i.e., a groupoid object in the category of Lie groupoids. Therefore, it has a natural categorical formulation. Compared to it, the abstract notion of a double Lie algebroid is rather complicated and non-obvious. The reason for this, is that properties of brackets for Lie algebroids are not expressed by diagrams, so one cannot approach double objects for them by methods of category theory. Mackenzie’s conditions given below came about as an abstraction of the properties of the double Lie algebroid of a double Lie groupoid discovered in [7], [8].

Definition 1. A double vector bundle

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \quad (1)$$

is a **double Lie algebroid** if all sides are (ordinary) Lie algebroids and the following **conditions I, II, and III** are satisfied:

- I.:** With respect to the vertical structures of Lie algebroids, $\begin{array}{c} D \\ \downarrow \\ A \end{array}$ and $\begin{array}{c} B \\ \downarrow \\ M \end{array}$, all maps related with the horizontal vector bundle structures are Lie algebroid morphisms (more precisely, this includes the projections, the zero sections, the fiberwise addition and multiplication by scalars). The same holds with vertical/horizontal structures interchanged.
- II.:** The horizontal arrows in the diagram

$$\begin{array}{ccc} D & \xrightarrow{a} & TB \\ \downarrow & & \downarrow \\ A & \xrightarrow{a} & TM \end{array}$$

where at the right there is the tangent prolongation of the Lie algebroid $B \rightarrow M$, and the horizontal arrows are the anchors, define a Lie algebroid morphism. The same holds with vertical/horizontal structures interchanged.

- III.:** The vertical arrows in the diagram

$$\begin{array}{ccc} D^{*A} & \longrightarrow & K^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

define a Lie algebroid morphism. Here K denotes the core. The same holds with vertical and horizontal, and A and B , interchanged. The vector bundles in duality $D^{*A} \rightarrow K^*$ and $D^{*B} \rightarrow K^*$ define a Lie bialgebroid.

A detailed analysis of these conditions will be given in the next sections. Now we wish to recall only the following. A *double vector bundle* such as (1) is defined by the condition that all vector bundle structure maps in one direction (horizontal or vertical) are vector bundle morphisms for the other direction. Its *core* K is defined as the intersection of the kernels of the projections $D \rightarrow A$ and $D \rightarrow B$ considered as vector bundle morphisms (w.r.t. the other structure). K is a vector bundle over M . It is a theorem due to Mackenzie [11] that taking the two duals of D considered as a vector bundle either over A or over B leads to two double vector bundles

$$\begin{array}{ccc} D^{*A} & \longrightarrow & K^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \quad \text{and} \quad \begin{array}{ccc} D^{*B} & \longrightarrow & B \\ \downarrow & & \downarrow \\ K^* & \longrightarrow & M \end{array}$$

such that the vector bundles $D^{*A} \rightarrow K^*$ and $D^{*B} \rightarrow K^*$ over the dual K^* of the core K are —unexpectedly— in a natural duality. All these facts, as well as the notion of the *tangent prolongation* of a Lie algebroid, can be found in [15, Ch. 9], see also [14, 16, 17]. *Lie bialgebroids* were introduced by Mackenzie and Xu [18]. Their theory was advanced by Y. Kosmann-Schwarzbach [4], who in particular gave a very handy form of the definition, which we use. See [15, Ch. 12].

2. LIE ALGEBROIDS AND DOUBLE VECTOR BUNDLES: SOME BACKGROUND

In this section we develop tools that will be later used for an alternative description of double Lie algebroids (our main goal).

Henceforth we work in the ‘super’ setup, i.e., we consider supermanifolds and bundles of supermanifolds. However, we systematically skip the prefix ‘super-’ except when we wish to make an emphasis. All the constructions from the previous section carry over to the super case.

We use *graded manifolds* as defined in [24], i.e., supermanifolds endowed with an extra \mathbb{Z} -grading in the algebras of functions, in general not related with parity. We refer to such grading as *weight*.

2.1. Lie algebroids and Lie antialgebroids. Let us recall some known facts concerning Lie algebroids.

It was first shown by Vaintrob [23] that Lie algebroids can be described by homological vector fields. We shall recall this correspondence using the description given in [24] in the language of derived brackets. As mentioned, we consider the ‘superized’ version (i.e., ‘super’ Lie algebroids) by default.

Let $F \rightarrow M$ be a vector bundle. The total space F is a graded manifold in a natural way, the (pullbacks of) functions on the base M having weight 0 and linear functions on the fibers, weight 1. Using weights is very helpful for describing various geometric objects. For example, vector fields of weight -1 on F correspond to sections of F (or ΠF , see below). Vector fields of weight 0 are generators of fiberwise linear transformations. Vector fields of weight 1 can be used to generate brackets of sections. More precisely: a **Lie antialgebroid** structure on $F \rightarrow M$, by definition, is given by a homological field $Q \in \text{Vect}(F)$ of weight 1.

There is a one-to-one correspondence between Lie antialgebroids and Lie algebroids, as follows.

Let Π denote the parity reversion functor and $F = \Pi E$ for a vector bundle $E \rightarrow M$. Then F is a Lie antialgebroid if and only if E is a Lie algebroid. The anchor and the bracket for the sections of E are given by the following formulas:

$$a(u)f := [[Q, i(u)], f] \quad (2)$$

and

$$i([u, v]) := (-1)^{\bar{u}} [[Q, i(u)], i(v)]. \quad (3)$$

Here $f \in C^\infty(M)$, and $u, v \in C^\infty(M, E)$ are sections. We use the natural odd injection

$$i: C^\infty(M, E) \rightarrow \text{Vect}(\Pi E), \quad (4)$$

which sends a section $u \in C^\infty(M, E)$ to the vector field $i(u) \in \text{Vect}(\Pi E)$ of weight -1 . The map i is an odd isomorphism between the space of sections $C^\infty(M, E)$ and the subspace $\text{Vect}_{-1}(\Pi E) \subset \text{Vect}(\Pi E)$ of all vector fields of weight -1 . By counting weights, one can see that the LHS's of (2) and (3) are well-defined. The properties of the bracket and anchor are deduced from the identity $Q^2 = 0$ as is standard in the derived brackets method. Conversely, starting from a Lie algebroid structure in $E \rightarrow M$, one can reconstruct Q on ΠE with the desired properties.

All these facts can be checked without coordinates; however, introducing local coordinates makes them particularly transparent. Let x^a denote local coordinates on the base M . We shall

use u^i and ξ^i for linear coordinates in the fibers of E and $F = \Pi E$, respectively. Changes of coordinates have the following form:

$$x^a = x^a(x'), \quad u^i = u^{i'} T_{i'}^i(x'),$$

and

$$\xi^i = \xi^{i'} T_{i'}^i(x').$$

The map $i: C^\infty(M, E) \rightarrow \text{Vect}(\Pi E)$ has the following appearance in coordinates²: if $u = u^i(x)e_i$, then

$$i(u) = (-1)^{\tilde{u}} u^i(x) \frac{\partial}{\partial \xi^i}. \quad (5)$$

Clearly, the RHS of (5) is the general form of a vector field of weight -1 on F . A vector field Q of weight 1 on F in coordinates has the form

$$Q = \xi^i Q_i^a(x) \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{ji}^k(x) \frac{\partial}{\partial \xi^k}.$$

Equations (2) and (3) produce the following formulas for the anchor:

$$a(u) = u^i(x) Q_i^a(x) \frac{\partial}{\partial x^a},$$

and for the brackets:

$$[u, v] = \left(u^i Q_i^a \partial_a v^k - (-1)^{\tilde{u}(\tilde{v}+1)} v^i Q_i^a \partial_a u^k - (-1)^{\tilde{i}(\tilde{v}+1)} u^i v^j Q_{ji}^k \right) e_k,$$

where we abbreviated $\partial_a = \partial/\partial x^a$. In particular, for the elements of the local frame e_i we have

$$[e_i, e_j] = (-1)^{\tilde{j}} Q_{ij}^k(x) e_k.$$

For the record, we shall also include explicit expressions specifying the Poisson and Schouten brackets induced on the total spaces of the vector bundles E^* and ΠE^* , respectively. Using the local coordinates u_i and ξ_i in the respective fibers (so that the bilinear forms $u^i u_i$ and $u^i \xi_i$ are invariant), we obtain

$$\{x^a, x^b\} = 0, \quad \{u_i, x^a\} = Q_i^a(x), \quad \{u_i, u_j\} = (-1)^{\tilde{j}} Q_{ij}^k(x) u_k$$

for the Poisson brackets of the coordinates, and

$$\{x^a, x^b\} = 0, \quad \{\xi_i, x^a\} = Q_i^a(x), \quad \{\xi_i, \xi_j\} = (-1)^{\tilde{j}} Q_{ij}^k(x) \xi_k$$

for the Schouten brackets. They appear exactly the same, but one should not forget that the Poisson bracket is even and the Schouten bracket is odd. For the sign conventions and other information see [24]. For a coordinate-free description of these Lie-Poisson and Lie-Schouten brackets (without ‘super’) see [15, Ch. 7 and Ch. 10].

Now let us proceed to double vector bundles.

²Here we denoted the components $u^i(x)$ of a section $u = u^i(x)e_i$ in the same way as the fiber coordinates u^i . To avoid confusion note that the parity of a component $u^i(x)$ is the same as that of the corresponding variable u^i if the section u is even and it is of the opposite parity if the section u is odd.

2.2. Parity reversions for double vector bundles. For double vector bundles see [15, Ch. 9], [14], and the recent paper [2]. All the necessary notions readily carry over to the supermanifold setup. In particular, this allows to consider parity reversions for double and multiple vector bundles. (Such operations should be studied together with the duality operations of Mackenzie theory [14] and [2].)

Let

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \quad (6)$$

be a double vector bundle (in the category of supermanifolds). The manifold D is naturally bi-graded, by weights corresponding to the two vector bundle structures. We denote these weights by w_1 and w_2 , or by w_A and w_B , as convenient.

A double vector bundle (6) allows **fiberwise reversions of parity** in both directions, horizontal and vertical. We denote the corresponding operations by Π_1 and Π_2 (or by Π_A and Π_B when convenient). The vertical reversion of parity $\Pi_1 = \Pi_A$ gives

$$\begin{array}{ccc} \Pi_A D & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \quad (7)$$

which is a new double vector bundle. One can apply the horizontal reversion of parity to it (applying the vertical reversion again takes us back), or do it the other way round. The parity reversions Π_1 and Π_2 and their compositions are covariant functors on the category of double vector bundles. (Recall that the morphisms in this category are the morphisms of diagrams (6) inducing fiberwise linear maps of all constituent ordinary vector bundles. In the language of graded manifolds, this means simply that a map of total spaces should preserve both weights w_1 and w_2 .)

Proposition 1. *The operations Π_1 and Π_2 commute:*

$$\Pi_1 \Pi_2 = \Pi_2 \Pi_1 ,$$

or, more precisely, there is a canonical isomorphism of functors

$$I_{12}: \Pi_2 \Pi_1 \rightarrow \Pi_1 \Pi_2 .$$

In greater detail, for each double vector bundle given by (6), there is an isomorphism of double vector bundles

$$\begin{array}{ccccc} \Pi_B \Pi_A D & \longrightarrow & \Pi B & & \Pi_A \Pi_B D & \longrightarrow & \Pi B \\ \downarrow & & \downarrow & \xrightarrow{I_{12}} & \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M & & \Pi A & \longrightarrow & M \end{array} \quad (8)$$

(where we used the more suggestive notation $\Pi_A = \Pi_1$ and $\Pi_B = \Pi_2$) commuting with the morphisms induced by all double vector bundles morphisms $\Phi: D_1 \rightarrow D_2$.

We can denote the common value of the ultimate total spaces by $\Pi^2 D$ (up to a natural isomorphism). If we need a particular choice of the double vector bundle, we can agree for concreteness that $\Pi^2 D = \Pi_2 \Pi_1 D$. We call $\Pi^2 D$ the *complete parity reversion* of D .

Remark 2.1. Everything extends immediately to the n -fold case, with $\Pi^n D$ being the *complete parity reversion* of the ultimate total space D of an n -fold vector bundle. There are partial parity reversion operations Π_r such that $\Pi_r \Pi_s = \Pi_s \Pi_r$ (equality means natural isomorphism) and $\Pi^n D = \Pi_n \cdot \dots \cdot \Pi_1 D$. We consider multiple vector bundles in Section 6.

We shall explain now the constructions and give a proof of Proposition 1 using local coordinates. The coordinate language is particularly handy for visualizing double (and multiple) vector bundles; it is as follows. Consider a double vector bundle given by (6). As above, denote local coordinates on M by x^a . Let u^i and w^α be linear coordinates on the fibers of $A \rightarrow M$ and $B \rightarrow M$, respectively. On D we have coordinates $x^a, u^i, w^\alpha, z^\mu$ so that u^i, z^μ are linear fiber coordinates for $D \rightarrow B$ and w^α, z^μ , for $D \rightarrow A$. Coordinate changes have the form:

$$x^a = x^a(x'), \quad (9)$$

$$u^i = u^{i'} T_{i'}^i(x'), \quad (10)$$

$$w^\alpha = w^{\alpha'} T_{\alpha'}^\alpha(x'), \quad (11)$$

$$z^\mu = z^{\mu'} T_{\mu'}^\mu(x') + u^{i'} w^{\alpha'} T_{\alpha' i'}^\mu(x'). \quad (12)$$

(Note that the submanifold specified by the equations $u^i = 0$ and $w^\alpha = 0$ is the core K of the double vector bundle D and the variables z^μ restricted to K become fiber coordinates for the vector bundle $K \rightarrow M$.) This is a convenient coordinate description of a double vector bundle structure. (The reader who lacks a taste for coordinates may translate everything into a language of local trivializations.) In particular, for two weights we have $\mathbf{w}_1 = \#u + \#z$ and $\mathbf{w}_2 = \#w + \#z$, where $\#$ denotes the degree in the respective variable. Everything extends directly to multiple vector bundles, see Section 6.

Partial parity reversions can now be described as follows.

For the double vector bundle given by (7), we have $x^a, u^i, \eta^\alpha, \theta^\mu$ as coordinates on $\Pi_A D$, so that $\widetilde{\eta}^\alpha = \widetilde{w}^\alpha + 1 = \widetilde{\alpha} + 1$, $\widetilde{\theta}^\mu = \widetilde{z}^\mu + 1 = \widetilde{\mu} + 1$, and the changes of coordinates are

$$\begin{aligned} u^i &= u^{i'} T_{i'}^i, \\ \eta^\alpha &= \eta^{\alpha'} T_{\alpha'}^\alpha, \\ \theta^\mu &= \theta^{\mu'} T_{\mu'}^\mu + (-1)^{i'} u^{i'} \eta^{\alpha'} T_{\alpha' i'}^\mu, \end{aligned}$$

where we suppress coordinates on M . Here η^α, θ^μ are fiber coordinates for $\Pi_A D \rightarrow A$ and η^α are fiber coordinates for $\Pi B \rightarrow M$. (Here and below, switching from Latin to Greek letters and back is meant to remind about changing of parity.)

Similarly, for

$$\begin{array}{ccc} \Pi_B D & \longrightarrow & B \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array} \quad (13)$$

we have $x^a, \xi^i, w^\alpha, \varepsilon^\mu$ as coordinates on $\Pi_B D$, where $\tilde{\xi}^i = \tilde{u}^i + 1 = \tilde{i} + 1$ and $\tilde{\varepsilon}^\mu = \tilde{z}^\mu + 1 = \tilde{\mu} + 1$, with the changes of coordinates

$$\begin{aligned}\xi^i &= \xi^{i'} T_{i'}^i, \\ w^\alpha &= w^{\alpha'} T_{\alpha'}^\alpha, \\ \varepsilon^\mu &= \varepsilon^{\mu'} T_{\mu'}^\mu + \xi^{i'} w^{\alpha'} T_{\alpha' i'}^\mu.\end{aligned}$$

By applying to the bundles $\Pi_A D$ and $\Pi_B D$ the parity reversions in the other directions, we obtain, respectively, the double vector bundle

$$\begin{array}{ccc}\Pi_B \Pi_A D & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M\end{array}\tag{14}$$

with coordinates $x^a, \xi^i, \eta^\alpha, t^\mu$ on $\Pi_B \Pi_A D$ and the transformation law

$$\begin{aligned}\xi^i &= \xi^{i'} T_{i'}^i, \\ \eta^\alpha &= \eta^{\alpha'} T_{\alpha'}^\alpha, \\ t^\mu &= t^{\mu'} T_{\mu'}^\mu + (-1)^{\tilde{i}'} \xi^{i'} \eta^{\alpha'} T_{\alpha' i'}^\mu,\end{aligned}$$

and the double vector bundle

$$\begin{array}{ccc}\Pi_A \Pi_B D & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M\end{array}\tag{15}$$

with coordinates $x^a, \xi^i, \eta^\alpha, s^\mu$ on $\Pi_A \Pi_B D$ and the transformation law

$$\begin{aligned}\xi^i &= \xi^{i'} T_{i'}^i, \\ \eta^\alpha &= \eta^{\alpha'} T_{\alpha'}^\alpha, \\ s^\mu &= s^{\mu'} T_{\mu'}^\mu + (-1)^{\tilde{i}'+1} \xi^{i'} \eta^{\alpha'} T_{\alpha' i'}^\mu.\end{aligned}$$

The explanation of the sign factors appearing before the second terms in the transformation laws for $\theta^\mu, \varepsilon^\mu, t^\mu$ and s^μ above is as follows. To calculate the transformation for the fiber coordinates in the parity-reversed vector bundle, we have to express the transformation law in the original bundle so that its fiber coordinates all stand at the left and then to replace them by the variables of the opposite parity (see, e.g., [24]). So, for example, for $\Pi_A D \rightarrow A$, we take $z^\mu = z^{\mu'} T_{\mu'}^\mu(x') + u^{i'} w^{\alpha'} T_{\alpha' i'}^\mu(x')$ and re-write it as $z^\mu = z^{\mu'} T_{\mu'}^\mu(x') + (-1)^{\tilde{i}'\tilde{\alpha}'} w^{\alpha'} u^{i'} T_{\alpha' i'}^\mu(x')$. By replacing w, z by η, θ , resp., we arrive at $\theta^\mu = \theta^{\mu'} T_{\mu'}^\mu(x') + (-1)^{\tilde{i}'\tilde{\alpha}'} \eta^{\alpha'} u^{i'} T_{\alpha' i'}^\mu(x')$, which, after re-arranging back to the initial order, gives $\theta^\mu = \theta^{\mu'} T_{\mu'}^\mu(x') + (-1)^{\tilde{i}'} u^{i'} \eta^{\alpha'} T_{\alpha' i'}^\mu(x')$ because the parities of $w^{\alpha'}$ and $\eta^{\alpha'}$ are opposite. A short-cut allowing to visualize this easily is to treat the passage to the linear coordinates of reversed parity as a formal multiplication (from the left) by an odd symbol Π satisfying the usual commutation relations with whatever coefficients (so that in our example η and θ are regarded as Πw and Πz).

Note that the transformation laws for the ξ - and η -coordinates are the same on $\Pi_B \Pi_A D$ and $\Pi_A \Pi_B D$ (that is why we used for them the same letters), and the transformation laws for the

t -coordinates on $\Pi_B \Pi_A D$ and the s -coordinates on $\Pi_A \Pi_B D$ differ only by a sign. We define a transformation of double vector bundles

$$I_{12}: \Pi_B \Pi_A D \rightarrow \Pi_A \Pi_B D$$

by the formulas $I_{12}^*(\xi^i) = \xi^i$, $I_{12}^*(\eta^\alpha) = \eta^\alpha$, and $I_{12}^*(s^\mu) = -t^\mu$. Clearly, it is well-defined and is an isomorphism. If we have an arbitrary morphism of double vector bundles $\Phi: D_1 \rightarrow D_2$, it induces morphisms $\Pi_1 \Pi_2 D_1 \rightarrow \Pi_1 \Pi_2 D_2$ and $\Pi_2 \Pi_1 D_1 \rightarrow \Pi_2 \Pi_1 D_2$ (here the indices of Π_i indicate the “directions” of partial parity reversions and the indices of D_1, D_2 are the labels of double vector bundles in consideration). In coordinates they are as follows. For the original bundles:

$$\begin{aligned} \Phi^*(x_2^{a_2}) &= x_2^{a_2}(x_1), \\ \Phi^*(u_2^{i_2}) &= u_1^{i_1} \Phi_{i_1}^{i_2}(x_1), \\ \Phi^*(w_2^{\alpha_2}) &= w_1^{\alpha_1} \Phi_{\alpha_1}^{\alpha_2}(x_1), \\ \Phi^*(z_2^{\mu_2}) &= z_1^{\mu_1} \Phi_{\mu_1}^{\mu_2}(x_1) + u_1^{i_1} w_1^{\alpha_1} \Phi_{\alpha_1 i_1}^{\mu_2}(x_1). \end{aligned}$$

For the induced morphisms of the parity reversed bundles we obtain from here, for $\Pi_1 \Pi_2 D_1 \rightarrow \Pi_1 \Pi_2 D_2$:

$$\begin{aligned} \Phi^*(\xi_2^{i_2}) &= \xi_1^{i_1} \Phi_{i_1}^{i_2}(x_1), \\ \Phi^*(\eta_2^{\alpha_2}) &= \eta_1^{\alpha_1} \Phi_{\alpha_1}^{\alpha_2}(x_1), \\ \Phi^*(s_2^{\mu_2}) &= s_1^{\mu_1} \Phi_{\mu_1}^{\mu_2}(x_1) + (-1)^{\tilde{i}_1+1} \xi_1^{i_1} \eta_1^{\alpha_1} \Phi_{\alpha_1 i_1}^{\mu_2}(x_1), \end{aligned}$$

and for $\Pi_2 \Pi_1 D_1 \rightarrow \Pi_2 \Pi_1 D_2$:

$$\begin{aligned} \Phi^*(\xi_2^{i_2}) &= \xi_1^{i_1} \Phi_{i_1}^{i_2}(x_1), \\ \Phi^*(\eta_2^{\alpha_2}) &= \eta_1^{\alpha_1} \Phi_{\alpha_1}^{\alpha_2}(x_1), \\ \Phi^*(t_2^{\mu_2}) &= t_1^{\mu_1} \Phi_{\mu_1}^{\mu_2}(x_1) + (-1)^{\tilde{i}_1} \xi_1^{i_1} \eta_1^{\alpha_1} \Phi_{\alpha_1 i_1}^{\mu_2}(x_1). \end{aligned}$$

(Strictly speaking, we have to use the notations such as $\Phi^{\Pi_1 \Pi_2}$ for the induced morphisms, but we abbreviate them just to Φ . The explanation for the signs is as above.) The transformation I_{12} , for each of the double vector bundles, maps (ξ, η, t) to (ξ, η, s) with $s = -t$. Therefore the diagram

$$\begin{array}{ccc} \Pi_2 \Pi_1 D_1 & \xrightarrow{\Phi} & \Pi_2 \Pi_1 D_2 \\ I_{12} \downarrow & & \downarrow I_{12} \\ \Pi_1 \Pi_2 D_1 & \xrightarrow{\Phi} & \Pi_1 \Pi_2 D_2 \end{array}$$

is commutative; i.e., we see that the transformation I_{12} commutes with the induced morphisms or is “natural” in the categorical sense (an isomorphism of functors). This completes a proof of Proposition 1. \square

Remark 2.2. Our choice of I_{12} is not the only possible. One could prefer to change the sign of the “side” coordinates such as ξ^i or η^α . Our choice is to keep the transformation I_{12} identical on the sides and inducing $-\text{id}$ on the core bundle. This is the isomorphism used in the main statement below.

3. MAIN STATEMENT

In this section we give our main statement, which is a characterization of Mackenzie's double Lie algebroids in terms of graded Q -manifolds. Proofs will be given in Sections 4 and 5.

Definition 2. A double vector bundle

$$\begin{array}{ccc} H & \longrightarrow & G \\ \downarrow & & \downarrow \\ F & \longrightarrow & M \end{array} \quad (16)$$

is a **double Lie antialgebroid** if the manifold H is endowed with two homological vector fields Q_1 and Q_2 of weights $(1, 0)$ and $(0, 1)$, respectively, such that

$$[Q_1, Q_2] = 0. \quad (17)$$

Remark 3.1. In our earlier text [25], it was stated erroneously that, by taking $Q_1 + Q_2$, a double antialgebroid structure can be further reduced to a single homological field Q of total weight 1. However, such a field Q decomposed according to the weights w_1 and w_2 would include terms of weights $(2, -1)$ and $(-1, 2)$ besides the 'correct' terms of weights $(1, 0)$ and $(0, 1)$. So we indeed need two fields Q_1 and Q_2 and cannot get away with a single condition on total weight.

Remark 3.2. An extension to *multiple Lie antialgebroids* is immediate. An n -fold **Lie antialgebroid** is an n -fold vector bundle, which therefore gives rise to an n -graded structure on its (ultimate) total space, endowed with n commuting homological fields Q_r , $r = 1, \dots, n$, on the total space of weights $(0, \dots, 1, \dots, 0)$, respectively. Here 1 stands at the r -th place, all other weights being zero. We shall elaborate this in Section 6.

Example 3.1. Let (E, E^*) be a Lie bialgebroid with base M . Then

$$\begin{array}{ccc} T^*\Pi E = T^*\Pi E^* & \longrightarrow & \Pi E^* \\ \downarrow & & \downarrow \\ \Pi E & \longrightarrow & M \end{array} \quad (18)$$

is a double vector bundle, as one can check. This is a superization of Mackenzie [9]. The natural diffeomorphism $T^*\Pi E = T^*\Pi E^*$ in the upper-left corner of (18) is a superized version [24] of a theorem of Mackenzie and Xu [18] extending a statement of Tulczyjew [22]. The Lie algebroid

structures in E and E^* give rise to homological fields on ΠE and ΠE^* , respectively. These two vector fields correspond to two functions ('linear Hamiltonians') on $T^*\Pi E = T^*\Pi E^*$, of weights $(1, 0)$ and $(0, 1)$. That (E, E^*) is a bialgebroid is equivalent to the commutativity of these Hamiltonians (due to Roytenberg [21], see also [24]). Therefore the corresponding Hamiltonian vector fields make the double vector bundle (18) a double Lie antialgebroid. We shall come back to this example in Section 6.

Theorem 1 (A Characterization of Double Lie Algebroids). *A double Lie algebroid structure in a double vector bundle such as (6) is equivalent to a double Lie antialgebroid structure in the complete parity reversion double vector bundle*

$$\begin{array}{ccc} \Pi^2 D & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array} \quad (19)$$

i.e., to a pair of commuting homological fields Q_1 and Q_2 of weights $(1, 0)$ and $(0, 1)$ on $\Pi^2 D$.

To appreciate the statement one may compare the three conditions in Definition 1 with simple equation (17).

Remark 3.3. We may consider $\Pi^2 D$ as $\Pi_B \Pi_A D$ or as $\Pi_A \Pi_B D$. The correspondence between the Lie algebroids and the Lie antialgebroids given by Theorem 1 does not depend on this choice. However, it is important that the isomorphism I_{12} defined in Proposition 1 is used for an identification of $\Pi_A \Pi_B D$ and $\Pi_B \Pi_A D$.

Let us show how homological vector fields on $\Pi^2 D$ as in the Theorem generate Lie algebroid structures on all sides of (6). As in our discussion of a single Lie algebroid above, everything can be formulated in a coordinate-free setting. However, using coordinates sheds some extra light and gets to the formulas quicker.

For the sake of concreteness, consider $\Pi^2 D = \Pi_B \Pi_A D$ with natural coordinates $x^a, \xi^i, \eta^\alpha, t^\mu$ thereon. Consider a homological vector field $Q_1 \in \text{Vect}(\Pi^2 D)$ of weight $(1, 0)$ and a homological vector field $Q_2 \in \text{Vect}(\Pi^2 D)$ of weight $(0, 1)$. They have the general forms

$$Q_1 = \xi^i Q_i^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{ji}^k \frac{\partial}{\partial \xi^k} + \left(\xi^i \eta^\alpha Q_{\alpha i}^\beta + t^\mu Q_\mu^\beta \right) \frac{\partial}{\partial \eta^\beta} + \left(\frac{1}{2} \xi^i \xi^j \eta^\alpha Q_{\alpha ji}^\lambda + \xi^i t^\mu Q_{\mu i}^\lambda \right) \frac{\partial}{\partial t^\lambda}, \quad (20)$$

and

$$Q_2 = \eta^\alpha Q_\alpha^a \frac{\partial}{\partial x^a} + \left(\eta^\alpha \xi^i Q_{i\alpha}^j + t^\mu Q_\mu^j \right) \frac{\partial}{\partial \xi^j} + \frac{1}{2} \eta^\alpha \eta^\beta Q_{\beta\alpha}^\gamma \frac{\partial}{\partial \eta^\gamma} + \left(\frac{1}{2} \eta^\alpha \eta^\beta \xi^i Q_{i\beta\alpha}^\lambda + \eta^\alpha t^\mu Q_{\mu\alpha}^\lambda \right) \frac{\partial}{\partial t^\lambda}, \quad (21)$$

dictated by their respective weights. All coefficients here are functions of x^a . Now, *due to the fact that the vector field Q_1 has weight 0 w.r.t. the vertical fiber coordinates, it admits partial parity reversion in this direction*, giving a vector field on $\Pi_B D$ with fiber coordinates $\xi^i, w^\alpha, \varepsilon^\mu$:

$$Q_1^\Pi = \xi^i Q_i^a \frac{\partial}{\partial x^a} + \left((-1)^{\tilde{i}} \xi^i w^\alpha Q_{\alpha i}^\beta + \varepsilon^\mu Q_\mu^\beta \right) \frac{\partial}{\partial w^\beta} + \frac{1}{2} \xi^i \xi^j Q_{ji}^k \frac{\partial}{\partial \xi^k} + \left(\frac{1}{2} (-1)^{\tilde{i}+\tilde{j}} \xi^i \xi^j w^\alpha Q_{\alpha ji}^\lambda + (-1)^{\tilde{i}} \xi^i \varepsilon^\mu Q_{\mu i}^\lambda \right) \frac{\partial}{\partial \varepsilon^\lambda} \quad (22)$$

(we have regrouped terms here). Similarly, Q_2 admits the vertical parity reversion, which gives a vector field on $\Pi_A D$ with fiber coordinates $u^i, \eta^\alpha, \theta^\mu$:

$$Q_2^\Pi = \eta^\alpha Q_\alpha^a \frac{\partial}{\partial x^a} + \left((-1)^{\tilde{\alpha}} \eta^\alpha u^i Q_{i\alpha}^j - \theta^\mu Q_\mu^j \right) \frac{\partial}{\partial u^j} + \frac{1}{2} \eta^\alpha \eta^\beta Q_{\beta\alpha}^\gamma \frac{\partial}{\partial \eta^\gamma} + \left(-(-1)^{\tilde{\alpha}+\tilde{\beta}} \frac{1}{2} \eta^\alpha \eta^\beta u^i Q_{i\beta\alpha}^\lambda + (-1)^{\tilde{\alpha}} \eta^\alpha \theta^\mu Q_{\mu\alpha}^\lambda \right) \frac{\partial}{\partial \theta^\lambda}. \quad (23)$$

(To find these vector fields, one has to write down the infinitesimal flows of the original fields, which will be linear in the corresponding directions, and apply the parity reversions to them.)

Both Q_1^Π and Q_2^Π are again homological fields. They define Lie antialgebroid structures on the vector bundles $\Pi_B D \rightarrow B$ and $\Pi_A D \rightarrow A$, which correspond to Lie algebroid structures on $D \rightarrow B$ and $D \rightarrow A$, respectively.

The restrictions of Q_1^Π and Q_2^Π on ΠA and ΠB , respectively, treated as submanifolds (zero sections) in $\Pi_B D$ and $\Pi_A D$ are tangent to these submanifolds and define the homological vector fields

$$Q_1^{(0)} = \xi^i Q_i^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{ji}^k \frac{\partial}{\partial \xi^k} \quad (24)$$

on ΠA and

$$Q_2^{(0)} = \eta^\alpha Q_\alpha^a \frac{\partial}{\partial x^a} + \frac{1}{2} \eta^\alpha \eta^\beta Q_{\beta\alpha}^\gamma \frac{\partial}{\partial \eta^\gamma} \quad (25)$$

on ΠB . This gives Lie algebroid structures on $A \rightarrow M$ and $B \rightarrow M$.

4. ANALYSIS OF MACKENZIE'S CONDITIONS

To prove Theorem 1, we shall go in the direction opposite to that in the previous section. (For this reason, we shall change our notation for the homological vector fields slightly, as the reader will see.) Consider a double vector bundle given by (1). Assume that all four sides are Lie algebroids and describe them by homological vector fields. Then we study conditions I, II and III of Definition 1 and see what they mean in terms of these fields.

Recall the notion of a *Lie algebroid morphism*. It is non-obvious for different bases. See [15, §4.3] for the definition. Instead of it, we shall use the following statement:

Proposition 2 (Văntrob). *Suppose $E_1 \rightarrow M_1$ and $E_2 \rightarrow M_2$ are Lie algebroids defined by homological vector fields $Q_1 \in \text{Vect } \Pi E_1$ and $Q_2 \in \text{Vect } \Pi E_2$. A vector bundle map*

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

is a Lie algebroid morphism if and only if the vector fields Q_1 and Q_2 are Φ^Π -related, where

$$\Phi^\Pi: \Pi E_1 \rightarrow \Pi E_2$$

is the induced map of the opposite vector bundles.

This statement first appeared, without proof, in [23]. It is equivalent to saying that $\Phi^\Pi: \Pi E_1 \rightarrow \Pi E_2$ is a morphism of Lie antialgebroids. This condition is much easier to handle than the original definition of Lie algebroid morphisms. Recall that vector fields on two (super)manifolds are *F-related* (or are *intertwined by F*), for a smooth map F , if on smooth functions $F^* \circ Y = X \circ F^*$. In terms of the local flows g_t and h_t generated by Y and X , this means that the map F intertwines the flows: $g_t \circ F = F \circ h_t$.

Let us introduce some notation. We are given Lie algebroid structures in the vector bundles $D \rightarrow B$ and $A \rightarrow M$, and in $D \rightarrow A$ and $B \rightarrow M$. Hence we have four homological vector fields: $Q_{DB} \in \text{Vect}(\Pi_B D)$, $Q_{AM} \in \text{Vect}(\Pi A)$, $Q_{DA} \in \text{Vect}(\Pi_A D)$ and $Q_{BM} \in \text{Vect}(\Pi B)$. In the notation from Section 2, the vector bundle $\Pi_B D \rightarrow B$ has fiber coordinates ξ^i, ε^μ . The vector field Q_{DB} should have weight 1 in these variables. In the same way, the vector bundle $\Pi A \rightarrow M$ has fiber coordinates ξ^i and the vector field Q_{AM} is of weight 1 in ξ^i . Similarly, Q_{DA}

has weight 1 in the variables η^α, θ^μ and Q_{BM} , in the variables η^α (the fiber coordinates for the vector bundles $\Pi_A D \rightarrow A$ and $\Pi B \rightarrow M$, respectively).

We shall now study conditions I, II and III one by one.

4.1. Condition I. Condition I is the easiest for analysis.

Consider for concreteness the horizontal algebroid structures. Condition I requires that all vertical structure maps: bundle projections, zero sections, fiberwise addition and multiplication by scalars, give morphisms of Lie algebroids. We have the following diagrams to analyze:

$$\begin{array}{ccc}
 D & \longrightarrow & B \\
 p \downarrow & & \downarrow p \\
 A & \longrightarrow & M
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 D & \longrightarrow & B \\
 i \uparrow & & \uparrow i \\
 A & \longrightarrow & M
 \end{array}$$

and

$$\begin{array}{ccc}
 D \times_A D & \longrightarrow & B \times_M B \\
 +_A \downarrow & & \downarrow +_M \\
 D & \longrightarrow & B
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 D & \longrightarrow & B \\
 t_A \downarrow & & \downarrow t_M \\
 D & \longrightarrow & B
 \end{array}$$

In the language of homological vector fields, we see that the flows generated by the vector fields Q_{DB} and Q_{AM} should commute with all the vertical structure maps above, more precisely, with the maps induced on the total spaces of the parity reversed horizontal vector bundles. Commuting with the projection p means that (the flow of) the vector field Q_{DB} acts fiberwise on the total space of $\Pi_B D \rightarrow \Pi A$ and induces on ΠA (the flow of) the vector field Q_{AM} . Hence Q_{AM} is completely determined by Q_{DB} . Consider the action of the flow of Q_{DB} on the fibers of $\Pi_B D \rightarrow \Pi A$. Commutativity with fiberwise multiplication by scalars, $t_A: \Pi_B D \rightarrow \Pi_B D$, and addition, $+_A: \Pi_B D \times_{\Pi A} \Pi_B D \rightarrow \Pi_B D$, means that the flow of Q_{DB} is fiberwise linear (over ΠA). This is equivalent to the vector field Q_{DB} having weight 0 w.r.t. fiber coordinates on $\Pi_B D \rightarrow \Pi A$. Commutativity with the zero section $i: \Pi A \rightarrow \Pi_B D$ then comes about automatically.

We may summarize: if the horizontal Lie algebroid structures are described by homological vector fields Q_{DB} and Q_{AM} , then *Condition I* of Definition 1 is equivalent to Q_{DB} having vertical weight 0 (its horizontal weight being 1) and Q_{AM} being the restriction of Q_{DB} to the base $\Pi A \subset \Pi_B D$.

In the same way, for the vector fields $Q_{DA} \in \text{Vect}(\Pi_A D)$ and $Q_{BM} \in \text{Vect}(\Pi B)$ describing vertical Lie algebroid structures, we obtain that Q_{DA} should have weight $(0, 1)$ on $\Pi_A D$ and Q_{BM} be its restriction to ΠB .

In coordinates we arrive at the following general expressions for Q_{DB} , Q_{AM} , Q_{DA} and Q_{BM} dictated by their weights. For the vector fields describing the horizontal Lie algebroid structures:

$$\begin{aligned}
 Q_{DB} = & \xi^i Q_i^a \frac{\partial}{\partial x^a} + \left(\xi^i w^\alpha Q_{\alpha i}^\beta + \varepsilon^\mu Q_\mu^\beta \right) \frac{\partial}{\partial w^\beta} \\
 & + \frac{1}{2} \xi^i \xi^j Q_{ji}^k \frac{\partial}{\partial \xi^k} + \left(\frac{1}{2} \xi^i \xi^j w^\alpha Q_{\alpha ji}^\lambda + \xi^i \varepsilon^\mu Q_{\mu i}^\lambda \right) \frac{\partial}{\partial \varepsilon^\lambda}, \quad (26)
 \end{aligned}$$

a vector field of weight $(1, 0)$ on $\Pi_B D$, and

$$Q_{AM} = \xi^i Q_i^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{ji}^k \frac{\partial}{\partial \xi^k}, \quad (27)$$

a vector field of weight 1 on ΠA , which is a restriction of Q_{DB} on ΠA . Similarly, for the vector fields describing the vertical Lie algebroid structures, we have the following general form:

$$Q_{DA} = \eta^\alpha Q_\alpha^a \frac{\partial}{\partial x^a} + (\eta^\alpha u^i Q_{i\alpha}^j + \theta^\mu Q_\mu^j) \frac{\partial}{\partial u^j} + \frac{1}{2} \eta^\alpha \eta^\beta Q_{\beta\alpha}^\gamma \frac{\partial}{\partial \eta^\gamma} + \left(\frac{1}{2} \eta^\alpha \eta^\beta u^i Q_{i\beta\alpha}^\lambda + \eta^\alpha \theta^\mu Q_{\mu\alpha}^\lambda \right) \frac{\partial}{\partial \theta^\lambda}, \quad (28)$$

and

$$Q_{BM} = \eta^\alpha Q_\alpha^a \frac{\partial}{\partial x^a} + \frac{1}{2} \eta^\alpha \eta^\beta Q_{\beta\alpha}^\gamma \frac{\partial}{\partial \eta^\gamma}. \quad (29)$$

Warning: formulas (26)–(29) so obtained are similar to formulas (22)–(25), but there is a different sign convention in the notation for the coefficients. We shall stick to the conventions as in (26)–(29) from now on. This choice of signs is explained by the wish to have simpler expressions for Q_{DB} and Q_{DA} arising as primary objects.

For the homological vector fields given by (26)–(29), we need to deduce further constraints corresponding to Mackenzie's Conditions II and III.

Remark 4.1. It is worth re-iterating that the vector fields Q_{DB} and Q_{DA} determining the horizontal and vertical Lie algebroid structures for D are defined on different supermanifolds $\Pi_B D$ and $\Pi_A D$. The crucial fact however is that each of them has weight zero in the second direction. This will allow us to apply an additional parity reversion and arrive finally at a pair of vector fields Q_1 and Q_2 defined on a common domain $\Pi^2 D$ (by using the theory developed in §2.2).

4.2. Condition II. Consider Condition II of Definition 1. For the diagram

$$\begin{array}{ccc} D & \xrightarrow{a} & TB \\ \downarrow & & \downarrow \\ A & \xrightarrow{a} & TM \end{array} \quad (30)$$

which is supposed to give a Lie algebroid morphism

$$\begin{array}{ccc} D & & TB \\ \downarrow & \longrightarrow & \downarrow \\ A & & TM \end{array} \quad (31)$$

we first need to explicate the tangent prolongation Lie algebroid $TB \rightarrow TM$. The definition is in [15, §9.7]. We shall use the following proposition allowing to work directly with the homological vector fields.

Proposition 3. *The tangent prolongation Lie algebroid $TE \rightarrow TM$ of a Lie algebroid $E \rightarrow M$ specified by a homological vector field $Q \in \text{Vect}(\Pi E)$ is given by the tangent prolongation vector field \hat{Q} , which is an (automatically homological) vector field on $T(\Pi E) = \Pi_{TM} TE$.*

Note that, for any vector bundle $E \rightarrow M$, taking tangents leads to a double vector bundle

$$\begin{array}{ccc} TE & \longrightarrow & E \\ \downarrow & & \downarrow \\ TM & \longrightarrow & M \end{array}$$

(see [15, §3.4]). In particular, partial parity reversions make sense.

Proposition 3 is a completely natural statement and should be known to experts; however, when we needed it in [25], we could not find it in the literature and had to check it ourselves. (In fact, the classical version — in terms of linear Poisson structures — was in [18, Theorem 5.6].³) For completeness, here we include a proof, which can go as follows.

Proof of Proposition 3. First, we express the bracket and the anchor for the tangent prolongation Lie algebroid in terms of a local basis of sections of TE over TM . We shall use the notation of section 2.1, so e_i is a local frame for $E \rightarrow M$ and u^i are the corresponding fiber coordinates. The induced local frame for $TE \rightarrow TM$ may be denoted (\bar{e}_i, \bar{e}_i) , so that $u^i \bar{e}_i + \dot{u}^i \bar{e}_i$ is invariant. In Mackenzie's notation, $\bar{e}_i = \hat{e}_i$; these elements correspond to the basis e_i of the core, which in this case is the vector bundle $E \rightarrow M$ itself. At the same time, we have $\bar{e}_i = T(e_i)$, the tangents of the sections e_i . The definition of the bracket of sections of $TE \rightarrow TM$ given in [15, §9.7] translates, for the basis sections, into the equations

$$[\bar{e}_i, \bar{e}_j] = Q_{ij}^k \bar{e}_k + \dot{Q}_{ij}^k \bar{e}_k, \quad [\bar{e}_i, \bar{e}_j] = Q_{ij}^k \bar{e}_k, \quad [\bar{e}_i, \bar{e}_j] = 0.$$

Here, to avoid extra signs, the formulas are shown for the case when E and M are ordinary manifolds. There is no problem to write them in the general case. The expressions such as \dot{Q}_i^a stand for $\dot{x}^b \partial_b Q_i^a$, etc. The definition of the anchor [15, §9.7] translates, similarly, into

$$a(\bar{e}_i) = Q_i^a \frac{\partial}{\partial x^a} + \dot{Q}_i^a \frac{\partial}{\partial \dot{x}^a}, \quad a(\bar{e}_i) = Q_i^a \frac{\partial}{\partial \dot{x}^a}.$$

Now, we can compare this with the tangent prolongation of the vector field on ΠE ,

$$Q = \xi^i Q_i^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{ji}^k \frac{\partial}{\partial \xi^k},$$

corresponding to the Lie algebroid structure of $E \rightarrow M$. The tangent prolongation \hat{Q} is

$$\begin{aligned} \hat{Q} = \xi^i Q_i^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{ji}^k \frac{\partial}{\partial \xi^k} \\ + \left(\dot{\xi}^i Q_i^a + \xi^i \dot{Q}_i^a \right) \frac{\partial}{\partial \dot{x}^a} + \left(\dot{\xi}^i \xi^j Q_{ji}^k + \frac{1}{2} \xi^i \xi^j \dot{Q}_{ji}^k \right) \frac{\partial}{\partial \dot{\xi}^k}. \end{aligned}$$

It is a vector field on $T(\Pi E) = \Pi_{TM} TE$. It is automatically homological because tangent prolongation maps the commutator of vector fields to the commutator of the prolongations. One can see immediately that the coefficients of this \hat{Q} define precisely the brackets and the anchor written above, which completes the proof. \square

Coming back to the analysis of Condition II, we see that by differentiating the field Q_{BM} given by (29), we obtain a vector field \hat{Q}_{BM} on $T(\Pi B) = \Pi_{TM} TB$,

$$\begin{aligned} \hat{Q}_{BM} = \eta^\alpha Q_\alpha^a \frac{\partial}{\partial x^a} + \frac{1}{2} \eta^\alpha \eta^\beta Q_{\beta\alpha}^\gamma \frac{\partial}{\partial \eta^\gamma} \\ + \left(\dot{\eta}^\alpha Q_\alpha^a + \eta^\alpha \dot{Q}_\alpha^a \right) \frac{\partial}{\partial \dot{x}^a} + \left(\dot{\eta}^\alpha \eta^\beta Q_{\beta\alpha}^\gamma + \frac{1}{2} \eta^\alpha \eta^\beta \dot{Q}_{\beta\alpha}^\gamma \right) \frac{\partial}{\partial \dot{\eta}^\gamma}, \quad (32) \end{aligned}$$

³ Compare with T. Courant [1, Theorem 6] where tangent Poisson structures were used for a definition. This paper also contains coordinate expressions for the tangent Lie algebroid structure.

which is, by Proposition 3, the homological vector field defining the tangent prolongation Lie algebroid in (31).

In order to simplify notation, we shall continue writing all the formulas in the rest of this section for the case when D, A, B, M are ordinary manifolds, not supermanifolds. This will allow us to avoid extra signs. Obviously, everything carries over to the general super case.

Recall the formula for the anchor map $a: D \rightarrow TB$. From (26) we can obtain

$$\dot{x}^a = u^i Q_i^a, \quad \dot{\eta}^\beta = u^i \eta^\alpha Q_{\alpha i}^\beta + \theta^\mu Q_\mu^\beta \quad (33)$$

for the corresponding map $\Pi_A D \rightarrow T(\Pi B)$.

We see that the condition that (31) is a morphism of Lie algebroids translates into the condition that the map given by (33) intertwines the vector fields Q_{DA} and \hat{Q}_{BM} given by (28) and (32). After simplification, this gives the following four equations:

$$Q_\mu^\beta Q_\beta^a = Q_\mu^j Q_j^a, \quad (34)$$

$$Q_{\alpha i}^\beta Q_\beta^a + Q_i^b \partial_b Q_\alpha^a = Q_\alpha^b \partial_b Q_i^a + Q_{i\alpha}^j Q_j^a, \quad (35)$$

$$Q_{[\alpha i}^\delta Q_{\beta] \delta}^\gamma + Q_i^b \partial_b Q_{\beta \alpha}^\gamma = Q_{[\alpha}^a \partial_a Q_{\beta] i}^\gamma + Q_{i[\alpha}^j Q_{\beta] j}^\gamma + Q_{\beta \alpha}^\delta Q_{\delta i}^\gamma + Q_{i\beta \alpha}^\lambda Q_\lambda^\gamma, \quad (36)$$

and

$$Q_\mu^\alpha Q_{\beta \alpha}^\gamma = -Q_\beta^a \partial_a Q_\mu^\gamma + Q_\mu^j Q_{\beta j}^\gamma - Q_{\mu \beta}^\lambda Q_\lambda^\gamma \quad (37)$$

(the square brackets denote alternation in the appropriate indices, e.g., $Q_{[\alpha i}^\delta Q_{\beta] \delta}^\gamma = Q_{\alpha i}^\delta Q_{\beta \delta}^\gamma - Q_{\beta i}^\delta Q_{\alpha \delta}^\gamma$).

To complete the analysis of Condition II, we have to consider diagrams similar to (30) and (31):

$$\begin{array}{ccc} D & \longrightarrow & B \\ a \downarrow & & \downarrow a \\ TA & \longrightarrow & TM \end{array} \quad \text{and} \quad \begin{array}{ccc} D & \longrightarrow & B \\ & & \downarrow \\ TA & \longrightarrow & TM \end{array}$$

with A and B interchanged. This adds two equations to the system (34)–(37):

$$Q_{[i\alpha}^l Q_{j]l}^k + Q_\alpha^b \partial_b Q_{ji}^k = Q_{[i}^a \partial_a Q_{j]\alpha}^k + Q_{\alpha[i}^\beta Q_{j]\beta}^k + Q_{ji}^l Q_{l\alpha}^k + Q_{\alpha ji}^\lambda Q_\lambda^k \quad (38)$$

and

$$Q_\mu^i Q_{ji}^k = -Q_j^a \partial_a Q_\mu^k + Q_\mu^\beta Q_{j\beta}^k - Q_{\mu j}^\lambda Q_\lambda^k. \quad (39)$$

The system of equations (34)–(39) is symmetric w.r.t. the exchange of A and B . (Note that each of equations (34) and (35) is symmetric w.r.t. this exchange. Equations (36) and (37) are transformed to equations (38) and (39), respectively.)

The system of equations (34)–(39) is equivalent to Condition II of Definition 1. Notice that it is bilinear in the vector fields Q_{DA} and Q_{DB} .

4.3. Condition III, and conclusion of the proof. Condition III requires “deciphering” more than the other conditions. It consists of two parts. First of all, we have to consider the diagrams:

$$\begin{array}{ccc} D^{*A} & \longrightarrow & K^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \quad (40)$$

and

$$\begin{array}{ccc} D^{*B} & \longrightarrow & B \\ \downarrow & & \downarrow \\ K^* & \longrightarrow & M \end{array} \quad (41)$$

and understand why the top horizontal arrow in (40) and the left vertical arrow in (41) are Lie algebroids with base K^* . This involves recalling some theory due to Mackenzie.

The *core* K of a double vector bundle (6) (see [15, Ch. 9]) is a vector bundle over the base M with fiber coordinates z^μ and the transformation law

$$z^\mu = z^{\mu'} T_{\mu'}^\mu(x')$$

obtained from (12) by setting u^i and w^α to zero.

When we dualize the vertical bundle $D \rightarrow A$ in (1), we obtain the bundle $D^{*A} \rightarrow A$ with fiber coordinates w_α, z_μ (with lower indices) so that the form $w^\alpha w_\alpha + z^\mu z_\mu$ giving the pairing in coordinates is invariant. We arrive at the following transformation laws:

$$w_{\alpha'} = T_{\alpha'}^\alpha w_\alpha + u^{i'} T_{\alpha' i'}^\mu z_\mu, \quad (42)$$

$$z_{\mu'} = T_{\mu'}^\mu z_\mu, \quad (43)$$

where $(T_{\mu'}^\mu)$ and $(T_{\mu}^{\mu'})$ are reciprocal matrices, and the transformation of u^i remains as in (10). This explains the double vector bundle structure of (40), in particular the vector bundle $D^{*A} \rightarrow K^*$. (Note that z_μ can be identified with fiber coordinates for $K^* \rightarrow M$.) The core of this new double vector bundle is $B^* \rightarrow M$.

The same holds when we dualize over B . The total space of the vector bundle $D^{*B} \rightarrow B$ has coordinates $x^a, u_i, w^\alpha, z_\mu$, the coordinates (u_i, z_μ) being dual to (u^i, z^μ) on D . Hence the transformation law

$$u_{i'} = T_{i'}^i u_i + w^{\alpha'} T_{\alpha' i'}^\mu z_\mu, \quad (44)$$

$$z_{\mu'} = T_{\mu'}^\mu z_\mu, \quad (45)$$

from which we immediately obtain the double vector bundle structure of (41). The new core is $A^* \rightarrow M$.

Treating D^{*A} and D^{*B} as vector bundles over K^* , with fiber coordinates (u^i, w_α) and (u_i, w^α) , respectively, we arrive at a surprising natural duality between them discovered in [11] (see also [14] and [15, §9.2]), with the pairing given by the bilinear form

$$u^i u_i - w^\alpha w_\alpha. \quad (46)$$

Here the minus sign between the two terms is absolutely essential for the invariance; due to it the terms with z_μ appearing in the change of coordinates cancel each other.

Remark 4.2. Unlike the minus sign between the two terms, a choice of a common sign in formula (46) cannot be fixed by invariance considerations. The form with the opposite sign $-u^i u_i + w^\alpha w_\alpha$ defines an equally good pairing of vector bundles. Therefore the ‘Mackenzie duality’ between the bundles $D^{*A} \rightarrow K^*$ and $D^{*B} \rightarrow K^*$ is canonical up to a sign. This fact does not really affect anything in our constructions.

Now, the Lie algebroid structures on the vector bundles $D^{*A} \rightarrow K^*$ and $D^{*B} \rightarrow K^*$ follow as a consequence of the two non-trivial facts: 1) the fiberwise linearity over the base K^* of the Poisson brackets on the total spaces D^{*A} and D^{*B} induced by the Lie algebroid structures on the vector bundles $D \rightarrow A$ and $D \rightarrow B$, respectively; and 2) the above duality between the vector bundles $D^{*A} \rightarrow K^*$ and $D^{*B} \rightarrow K^*$. By this duality, the linear Poisson structure on each bundle $D^{*A} \rightarrow K^*$ and $D^{*B} \rightarrow K^*$ induces a Lie algebroid structure on the bundle $D^{*B} \rightarrow K^*$ and $D^{*A} \rightarrow K^*$, respectively (uniquely up to a sign).

This can be readily expressed in our language.

In order to obtain the homological vector fields specifying the Lie algebroid structures for $D^{*B} \rightarrow K^*$ and $D^{*A} \rightarrow K^*$, one can follow these steps: starting from the homological vector field Q_{DA} on $\Pi_A D$ construct the Lie-Poisson bracket on D^{*A} ; use the linearity of this bracket over K^* to obtain the Lie algebroid structure for the dual bundle $(D^{*A})^{*K^*} \rightarrow K^*$; apply the duality between $D^{*A} \rightarrow K^*$ and $D^{*B} \rightarrow K^*$ to re-write that as a Lie algebroid structure on $D^{*B} \rightarrow K^*$, and finally obtain the corresponding homological vector field on $\Pi_{K^*} D^{*B}$, which we denote Q_{DA}^* . (Similarly, with A and B interchanged.) Alternatively, the following shortcut argument may be used. Since the homological vector field Q_{DA} on $\Pi_A D$ defining the vertical Lie algebroid structures of the original double vector bundle has weight 0 in the horizontal direction, it generates fiberwise linear transformations and therefore allows the ‘transpose’ in that direction. More precisely, one has to take the contragredient transformation (the adjoint of the inverse) for the corresponding linear flow and the generator of the new flow is the desired (automatically homological) vector field Q_{DA}^* on $\Pi_{K^*} D^{*B}$.

In coordinates $x^a, \xi_i, \eta^\alpha, z_\mu$ on $\Pi_{K^*} D^{*B}$, we obtain the following expression for the homological vector field Q_{DA}^* :

$$Q_{DA}^* = \eta^\alpha Q_\alpha^a \frac{\partial}{\partial x^a} + \left(\xi_j \eta^\alpha Q_{i\alpha}^j - \frac{1}{2} \eta^\alpha \eta^\beta z_\lambda Q_{i\beta\alpha}^\lambda \right) \frac{\partial}{\partial \xi_i} + \frac{1}{2} \eta^\alpha \eta^\beta Q_{\beta\alpha}^\gamma \frac{\partial}{\partial \eta^\gamma} + (\xi_j Q_\mu^j - \eta^\alpha z_\lambda Q_{\mu\alpha}^\lambda) \frac{\partial}{\partial z_\mu}. \quad (47)$$

It defines the Lie algebroid structure on $D^{*B} \rightarrow K^*$.

In the same way we calculate the homological vector field Q_{DB}^* on $\Pi_{K^*} D^{*A}$ giving the Lie algebroid structure on $D^{*A} \rightarrow K^*$. In coordinates $x^a, \xi^i, \eta_\alpha, z_\mu$, we have

$$Q_{DB}^* = \xi^i Q_i^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{ji}^k \frac{\partial}{\partial \xi^k} + \left(-\xi^i \eta_\beta Q_{\alpha i}^\beta - \frac{1}{2} \xi^i \xi^j z_\lambda Q_{\alpha ji}^\lambda \right) \frac{\partial}{\partial \eta_\alpha} + (\eta_\beta Q_\mu^\beta - \xi^i z_\lambda Q_{\mu i}^\lambda) \frac{\partial}{\partial z_\mu}. \quad (48)$$

One can immediately see that the vector field Q_{DA}^* on $\Pi_{K^*} D^{*B}$ and Q_{DB}^* on $\Pi_{K^*} D^{*A}$ are related by the projection. Hence the horizontal arrows in (41) give a Lie algebroid morphism. The same

is true for the vector fields Q_{DB}^* and Q_{AM} , and for the vertical arrows in (40). *We see that the first part of Condition III holds automatically!*

Let us examine the second part of Condition III, that **the dual bundles $D^{*A} \rightarrow K^*$ and $D^{*B} \rightarrow K^*$ with the described Lie algebroid structures form a Lie bialgebroid**. *It turns out to be the main condition.*

In our language it goes as follows. If we have two vector bundles in duality, a Lie algebroid structure on one of them induces the Lie-Schouten bracket on the total space of the parity-reversed other bundle, as we have recalled in §2.1. If both bundles are endowed with Lie algebroid structures, the condition that they make a Lie bialgebroid⁴ is equivalent to that for each parity-reversed bundle the homological vector field is a derivation of the Schouten bracket (see [4], [15, §12.1], also [21] and [24]). The condition is self-dual and it suffices to check just one bundle. In our case, there are the homological vector fields, Q_{DA}^* and Q_{DB}^* , on $\Pi_{K^*} D^{*B}$ and $\Pi_{K^*} D^{*A}$ respectively, and there are the Schouten brackets induced on each bundle by duality.

Consider the bundle $\Pi_{K^*} D^{*B}$ for concreteness. On $\Pi_{K^*} D^{*B}$ we obtain the following explicit expression for the Schouten brackets between the coordinates:

$$\begin{aligned} \{x^a, x^b\} &= \{x^a, z_\mu\} = \{z_\mu, z_\nu\} = 0, \\ \{\xi_i, x^a\} &= Q_i^a, \quad \{\xi_i, z_\mu\} = -Q_{\mu i}^\lambda z_\lambda, \\ \{\eta^\alpha, x^a\} &= 0, \quad \{\eta^\alpha, z_\mu\} = -Q_\mu^\alpha, \\ \{\xi_i, \xi_j\} &= Q_{ij}^k \xi_k + Q_{\alpha ij}^\lambda z_\lambda \eta^\alpha, \quad \{\xi_i, \eta^\alpha\} = Q_{\beta i}^\alpha \eta^\beta, \\ \{\eta^\alpha, \eta^\beta\} &= 0. \end{aligned} \tag{49}$$

(In the same way we can find explicitly the Schouten bracket induced on $\Pi_{K^*} D^{*A}$, which we skip.)

The second part of Mackenzie's Condition III amounts therefore to the condition that the vector field (47) is a derivation of the odd bracket (49).

Remark 4.3. Since the duality between the vector bundles $D^{*A} \rightarrow K^*$ and $D^{*B} \rightarrow K^*$ is defined up to a sign, the Lie algebroid structures for them are also defined up to common signs, and so are the homological vector field (47) and the Schouten bracket (49). Obviously, the compatibility condition is independent of these choices.

By a direct calculation, the derivation property for the field (47) w.r.t. the bracket (49) expands to the following system of nine equations:

$$Q_\alpha^a Q_\mu^\alpha - Q_\mu^i Q_i^a = 0 \tag{50}$$

$$-Q_\mu^i Q_{\nu i}^\lambda + Q_\mu^\alpha Q_{\nu \alpha}^\lambda - Q_\nu^i Q_{\mu i}^\lambda + Q_\nu^\alpha Q_{\mu \alpha}^\lambda = 0 \tag{51}$$

$$Q_{\alpha j}^\beta Q_\beta^a + Q_j^b \partial_b Q_\alpha^a - Q_{j \alpha}^i Q_i^a = Q_\alpha^b \partial_b Q_j^a \tag{52}$$

$$Q_j^a \partial_a Q_\mu^i + Q_\mu^k Q_{jk}^i - Q_\mu^\alpha Q_{j \alpha}^i = -Q_{\mu j}^\lambda Q_\lambda^i \tag{53}$$

$$\begin{aligned} Q_\mu^i Q_{\beta j i}^\lambda - Q_{\beta j}^\alpha Q_{\mu \alpha}^\lambda + Q_{\nu j}^\lambda Q_{\mu \beta}^\nu - Q_j^a \partial_a Q_{\mu \beta}^\lambda + Q_{\mu i}^\lambda Q_{j \beta}^i - Q_\mu^\alpha Q_{j \beta \alpha}^\lambda \\ = -Q_\beta^a \partial_a Q_{\mu j}^\lambda + Q_{\mu j}^\nu Q_{\nu \beta}^\lambda \end{aligned} \tag{54}$$

$$-Q_\mu^i Q_{\alpha i}^\gamma + Q_\lambda^\gamma Q_{\mu \alpha}^\lambda - Q_\mu^\beta Q_{\beta \alpha}^\gamma = -Q_\alpha^a \partial_a Q_\mu^\gamma \tag{55}$$

⁴More precisely, two Lie bialgebroids in duality.

$$Q_{\alpha j}^{\beta} Q_{i\beta}^k + Q_{jl}^k Q_{i\alpha}^l + Q_j^a \partial_a Q_{i\alpha}^k - Q_{\alpha i}^{\beta} Q_{j\beta}^k - Q_{il}^k Q_{j\alpha}^l - Q_i^a \partial_a Q_{j\alpha}^k \\ = Q_{\alpha}^a \partial_a Q_{ij}^k - Q_{ij}^l Q_{l\alpha}^k - Q_{\alpha ij}^{\mu} Q_{\mu}^k \quad (56)$$

$$Q_{\alpha ik}^{\lambda} Q_{j\beta}^k - Q_{\beta ik}^{\lambda} Q_{j\alpha}^k - Q_{\alpha i}^{\gamma} Q_{j\beta\gamma}^{\lambda} + Q_{\beta i}^{\gamma} Q_{j\alpha\gamma}^{\lambda} + Q_{\mu i}^{\lambda} Q_{j\beta\alpha}^{\mu} - Q_i^a \partial_a Q_{j\beta\alpha}^{\lambda} \\ - Q_{\alpha jk}^{\lambda} Q_{i\beta}^k + Q_{\beta jk}^{\lambda} Q_{i\alpha}^k + Q_{\alpha j}^{\gamma} Q_{i\beta\gamma}^{\lambda} - Q_{\beta j}^{\gamma} Q_{i\alpha\gamma}^{\lambda} - Q_{\mu j}^{\lambda} Q_{i\beta\alpha}^{\mu} + Q_j^a \partial_a Q_{i\beta\alpha}^{\lambda} \\ = -Q_{ij}^k Q_{k\beta\alpha}^{\lambda} - Q_{\mu\alpha}^{\lambda} Q_{\beta ij}^{\mu} + Q_{\mu\beta}^{\lambda} Q_{\alpha ij}^{\mu} \\ + Q_{\alpha}^a \partial_a Q_{\beta ij}^{\lambda} - Q_{\beta}^a \partial_a Q_{\alpha ij}^{\lambda} + Q_{\gamma ij}^{\lambda} Q_{\beta\alpha}^{\gamma}, \quad (57)$$

$$Q_{\beta k}^{\gamma} Q_{j\alpha}^k - Q_{\alpha k}^{\gamma} Q_{j\beta}^k + Q_{j\beta\alpha}^{\lambda} Q_{\lambda}^{\gamma} - Q_{\alpha j}^{\varepsilon} Q_{\beta\varepsilon}^{\gamma} + Q_{\beta j}^{\varepsilon} Q_{\alpha\varepsilon}^{\gamma} - Q_j^a \partial_a Q_{\beta\alpha}^{\gamma} \\ = -Q_{\beta\alpha}^{\varepsilon} Q_{\varepsilon j}^{\gamma} - Q_{\alpha}^a \partial_a Q_{\beta j}^{\gamma} + Q_{\beta}^a \partial_a Q_{\alpha j}^{\gamma} \quad (58)$$

Therefore, *Condition III of Definition 1 is analytically expressed by the system of equations (50)–(58), bilinear in the components of the original homological vector fields Q_{DA} and Q_{DB} .*

Notice that equations (50), (52), (53), (55), (56) and (58) in this system are already familiar. They are equivalent to equations (34), (35), (39), (37), (38) and (36), respectively, which together make an analytic expression of Condition II. *Hence, Condition III contains Condition II*⁵. Since Condition I of the definition of a double Lie algebroid is encoded in the forms of the homological vector fields specifying the horizontal and vertical Lie algebroid structures, Condition II is subsumed by Condition III, and Condition III is expressed by the system of equations (50)–(58) for the components of these fields, we conclude that **Mackenzie's definition of a double Lie algebroid reduces to this system of equations (50)–(58)**. This is the outcome of our analysis.

Note that this system is symmetric w.r.t. swapping of A and B as it should. (In more detail, equations (53) and (55) are exchanged under the transposition of A and B , and so are equations (56) and (58); each of the remaining five equations is symmetric itself.)

To finish the proof of Theorem 1 it remains to compare the system (50)–(58) with the commutativity condition for the vector fields $Q_1 = Q_{DB}^{\Pi}$ and $Q_2 = Q_{DA}^{\Pi}$ on $\Pi^2 D$ obtained from Q_{DB} and Q_{DA} by partial parity reversions:

$$Q_1 = \xi^i Q_i^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j Q_{ji}^k \frac{\partial}{\partial \xi^k} + \left(\xi^i \eta^{\alpha} Q_{\alpha i}^{\beta} + t^{\mu} Q_{\mu}^{\beta} \right) \frac{\partial}{\partial \eta^{\beta}} + \\ \left(\frac{1}{2} \xi^i \xi^j \eta^{\alpha} Q_{\alpha ji}^{\lambda} + \xi^i t^{\mu} Q_{\mu i}^{\lambda} \right) \frac{\partial}{\partial t^{\lambda}}, \quad (59)$$

and

$$Q_2 = \eta^{\alpha} Q_{\alpha}^a \frac{\partial}{\partial x^a} + (\eta^{\alpha} \xi^i Q_{i\alpha}^j - t^{\mu} Q_{\mu}^j) \frac{\partial}{\partial \xi^j} + \frac{1}{2} \eta^{\alpha} \eta^{\beta} Q_{\beta\alpha}^{\gamma} \frac{\partial}{\partial \eta^{\gamma}} + \\ \left(-\frac{1}{2} \eta^{\alpha} \eta^{\beta} \xi^i Q_{i\beta\alpha}^{\lambda} + \eta^{\alpha} t^{\mu} Q_{\mu\alpha}^{\lambda} \right) \frac{\partial}{\partial t^{\lambda}}. \quad (60)$$

These formulas are written in the coordinates $x^a, \xi^i, \eta^{\alpha}, t^{\mu}$ on $\Pi^2 D$, see the discussion in §2.2. Note that we use the isomorphism I_{12} (as defined in the proof of Proposition 1) for the identification of $\Pi_A \Pi_B D$ and $\Pi_B \Pi_A D$. (Eqs. (59) and (60) are similar to (20) and (21), with modified

⁵After this fact was first discovered in [25], Mackenzie gave for it a different proof within his original framework [16]. See also [17].

signs, see the remark after Eq. (29). Note also that throughout this section we have worked with a purely even D , for simplicity, so some particular signs have not shown up. All the calculations carry over to the general case of course.)

The calculation of the commutator of Q_1 and Q_2 is straightforward and shows that the commutativity relation

$$[Q_1, Q_2] = 0$$

expands to a system of equations that precisely coincides with (50)–(58).

Hence we conclude that *Mackenzie's Definition 1 is equivalent to the commutativity of the homological fields Q_1 and Q_2* , QUOD ERAT DEMONSTRANDUM.

5. THE BIG PICTURE

After presenting a ‘computational’ proof of our main statement, we shall now give a conceptual explanation. The argument in this section provides an alternative proof of Theorem 1, almost without calculations.

Let us again consider a double vector bundle

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \quad (61)$$

We shall assume that all sides of it have Lie algebroid structures and that each of these structures is compatible with the linear structure in the other direction. Thus we assume the obvious part of the definition of a double Lie algebroid (i.e., Condition I). As we have seen, this is equivalent to saying that the Lie algebroid structures on $D \rightarrow A$ and $D \rightarrow B$ are defined by homological vector fields of weights $(0, 1)$ and $(1, 0)$ on the ultimate total spaces of

$$\begin{array}{ccc} \Pi_A D & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \quad \text{and} \quad \begin{array}{ccc} \Pi_B D & \longrightarrow & B \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array} \quad (62)$$

respectively.

We now show how a compatibility condition for these two Lie algebroid structures can be introduced.

Let us return for a moment to ordinary Lie algebroids (or just Lie algebras). Suppose $E \rightarrow M$ is a vector bundle. It has three *neighbors*: the dual bundle E^* , the opposite bundle ΠE and the antidual ΠE^* . A Lie algebroid structure in E (which is a structure on the module of sections) is equivalently expressed by each of the following structures on its neighbors: a homological vector field of weight 1 on ΠE , a linear Poisson bracket on E^* , and a linear Schouten bracket on ΠE^* . The axioms of a Lie algebroid are contained in the equation $Q^2 = 0$ or in the Jacobi identities for the Poisson or Schouten bracket. The structures on E^* , ΠE and ΠE^* are structures on the total spaces⁶.

Acting in a similar way, let us consider all the neighbors of our double vector bundle (61). There are four operations that can be applied: vertical dual, horizontal dual, vertical reversion of

⁶That is, on the algebras of functions as opposed to a structure on sections of a vector bundle or on elements of a vector space.

parity, and horizontal reversion of parity. Besides (62) one obtains the following double vector bundles, which are the neighbors of (61).

The complete parity reversion of (61):

$$\begin{array}{ccc} \Pi^2 D & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array} \quad (63)$$

The two duals of (61):

$$\begin{array}{ccc} D^{*A} & \longrightarrow & K^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \quad \text{and} \quad \begin{array}{ccc} D^{*B} & \longrightarrow & B \\ \downarrow & & \downarrow \\ K^* & \longrightarrow & M \end{array} \quad (64)$$

The parity reversions of each of the duals:

$$\begin{array}{ccc} \Pi_A D^{*A} & \longrightarrow & \Pi K^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \quad \begin{array}{ccc} \Pi_B D^{*B} & \longrightarrow & B \\ \downarrow & & \downarrow \\ \Pi K^* & \longrightarrow & M \end{array} \quad (65)$$

$$\begin{array}{ccc} \Pi_{K^*} D^{*A} & \longrightarrow & K^* \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array} \quad \begin{array}{ccc} \Pi_{K^*} D^{*B} & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ K^* & \longrightarrow & M \end{array} \quad (66)$$

and

$$\begin{array}{ccc} \Pi^2 D^{*A} & \longrightarrow & \Pi K^* \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array} \quad \begin{array}{ccc} \Pi^2 D^{*B} & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ \Pi K^* & \longrightarrow & M \end{array} \quad (67)$$

This is the full list up to natural isomorphisms. These twelve objects, including the original double vector bundle (61), can be arranged into a four-valent colored graph (where from each vertex emanate two edges corresponding to taking the duals and two edges corresponding to the parity reversions).

Remark 5.1. In the multiple case, for an n -fold vector bundle, the number of edges emanating from each vertex is $n + n = 2n$.

The Lie algebroid structures on the sides of (61) obeying the linearity conditions, which were expressed above in terms of weights, generate a pair of structures for each of the neighbors (62)–(67), in various combinations. More symmetrically, we may say that each pair of structures for a particular double vector bundle from (61)–(67) is just a manifestation of one ‘pre-double structure’ — that is, without a compatibility condition yet. (All pairs contain the same information.) One can make a list of such structures. The next step will be to look for suitable compatibility conditions for each pair. The philosophy is that one should look for pairs where a compatibility

condition is formulated naturally, and take it as the definition of compatibility for an equivalent pair where such a condition does not come about in an obvious way.

In other words: suppose we do not know what a double Lie algebroid is; to get the right notion of compatibility for the Lie algebroids on the sides of the double vector bundle (61), examine its neighbors.

Our philosophy is that a verifiable compatibility condition lives on the total space.

Let us start from (64). For each of the double vector bundles in (64), the ultimate total space is a Lie algebroid (over the base K^*) and simultaneously possesses a linear Poisson bracket (linear over both bases). That means that the dual bundle over K^* is also a Lie algebroid, and one may ask whether they form a Lie bialgebroid. As the analysis of the previous section shows, this may be considered as the **Mackenzie definition** of a double Lie algebroid for (61), since it subsumes all his other conditions⁷.

Note that for Lie bialgebroids and Lie bialgebras, unlike Lie algebras or Lie algebroids equivalently manifesting themselves on a total space in terms of a linear Poisson bracket, or a linear Schouten bracket, or a homological field of weight 1, everything remarkably boils down to just one type of structure: namely, a QS -structure, i.e., a homological vector field and a Schouten bracket linked by the derivation condition (see [24]).

Now look at the neighbors (61)–(67). Among them *there are precisely five cases where a structure is induced on the total space*: (63), and the four bundles in (66), (67).

On the total space of each of the double vector bundles in (66) there is a Schouten bracket of weight $(-1, -1)$ and a homological vector field of weight $(0, 1)$ or $(1, 0)$, respectively. On the total space of the bundles in (67) there is a Poisson bracket of weight $(-1, -1)$ and a homological vector field of weight $(0, 1)$ or $(1, 0)$. A compatibility condition in each case is the derivation property of the vector field w.r.t. the bracket.

Proposition 4. *The compatibility conditions for the four bundles in (66) and (67) are equivalent, and are but different ways of saying that (D^{*A}, D^{*B}) is a Lie bialgebroid over K^* .*

Proof. Indeed, for any Lie bialgebroid (E, E^*) the compatibility can be stated in terms of either E or E^* (as a QS -structure on either ΠE or ΠE^* , respectively). This corresponds to one of the bundles in (66). In our special situation there is also an extra option of changing parity in the other direction (and turning a QS -structure into a QP -structure), which adds two equivalent descriptions in terms of the bundles in (67). \square

The remaining case is the total space of (63) where there are two homological vector fields of weights $(0, 1)$ and $(1, 0)$. A compatibility condition for them is of course commutativity.

We see now that there are essentially two conditions to compare: the Mackenzie bialgebroid condition, which lives on one of the bundles in (66), (67), and our commutativity relation for the bundle (63).

Proposition 5. *The Mackenzie bialgebroid condition and the commutativity of the two homological vector fields for (63) are equivalent.*

Proof. Consider one of the manifestations of the bialgebroid condition, say, for concreteness, on the first double vector bundle in (66). The derivation property means that the flow of the vector field preserves the Schouten bracket. On the other hand, the commutativity condition for (63)

⁷As already noted, after this fact was discovered [25], Mackenzie proved it directly in [16], see also [17], which makes the analysis in this section entirely independent of our coordinate calculations in Section 4.

means that the flow of one field preserves the other. Now the claim follows by functoriality: a linear transformation preserves a Lie bracket if and only if the adjoint or anti-adjoint map preserves the corresponding linear Poisson or Schouten bracket and if and only if the ‘ Π -symmetric’ map preserves the corresponding homological vector field. \square

Propositions 4 and 5 together imply Theorem 1.

6. APPLICATIONS AND GENERALIZATIONS

In this section we revise ‘Drinfeld doubles’ for Lie bialgebroids, introduce systematically multiple Lie algebroids already touched upon in the previous sections, and discuss how Lie bialgebroid theory may be extended to the multiple case.

6.1. Doubles of Lie bialgebroids. We already mentioned in the introduction that the problem of a Drinfeld double of a Lie bialgebroid was one of the motivations and a testing case for Mackenzie’s definition of double Lie algebroids.

Recall that Drinfeld’s *classical double* of a Lie bialgebra is again a Lie bialgebra with “good” properties. An analog of this construction for Lie bialgebroids turned out to be a puzzle. Three constructions of a ‘double’ have been suggested. Suppose (E, E^*) is a Lie bialgebroid over a base M . Liu, Weinstein and Xu [6] suggested to consider as its double a structure of a Courant algebroid on the direct sum $E \oplus E^*$. Mackenzie in [9], [10], [12] and Roytenberg in [21] suggested two different constructions based on the cotangent bundles T^*E and $T^*\Pi E$, respectively.

Roytenberg showed [21] that the Liu–Weinstein–Xu double can be recovered from his own construction using derived brackets, by generalizing the results of C. Roger [20] and Y. Kosmann-Schwarzbach [3], [5] for Lie bialgebras. Thus only the constructions of [21] and [9], [10] have to be compared.

Though approaches of [21] and [9], [10] look very different, we shall now establish their equivalence.

Both Roytenberg’s and Mackenzie’s construction use the statement that the cotangent bundles of dual vector bundles are isomorphic [18]. There is a double vector bundle

$$\begin{array}{ccc} T^*E = T^*E^* & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \quad (68)$$

Mackenzie [9] shows that it is a double Lie algebroid, which he calls the ‘cotangent double’ of a Lie bialgebroid (E, E^*) . He uses his original definition of double Lie algebroids and we do not need to elaborate his argument here.

On the other hand, Roytenberg [21] considers the diagram

$$\begin{array}{ccc} T^*\Pi E = T^*\Pi E^* & \longrightarrow & \Pi E^* \\ \downarrow & & \downarrow \\ \Pi E & \longrightarrow & M \end{array} \quad (69)$$

His further construction is as follows. Suppose $Q_E \in \text{Vect}(\Pi E)$ and $Q_{E^*} \in \text{Vect}(\Pi E^*)$ are the homological vector fields defining the Lie algebroid structures on $E \rightarrow M$ and $E^* \rightarrow M$, respectively. Assign to them the fiberwise linear functions H_E and H_{E^*} on the cotangent bundles $T^*\Pi E$ and $T^*\Pi E^*$, respectively. One proves [21] that under the natural symplectomorphism

$T^*\Pi E \rightarrow T^*\Pi E^*$ the linear function H_{E^*} on $T^*\Pi E^*$ corresponding to the vector field Q_{E^*} is transformed into the fiberwise quadratic function S_E on $T^*\Pi E$ specifying the Schouten bracket on ΠE induced by the Lie structure on E^* . Therefore the derivation property of Q_E w.r.t. the Schouten bracket on ΠE , which is the most convenient definition of a Lie bialgebroid [4] is equivalent to the commutativity of the Hamiltonians H_E and H_{E^*} under the canonical Poisson bracket. They generate commuting homological vector fields X_{H_E} and $X_{H_{E^*}}$ on the cotangent bundle $T^*\Pi E$. It was suggested in [21] to consider the sum $Q = X_{H_E} + X_{H_{E^*}}$, which is a homological field of total weight $+1$ on the graded manifold $T^*\Pi E$ as the desired ‘double’.

If we bear in mind that $T^*\Pi E$ is a double vector bundle and check that X_{H_E} and $X_{H_{E^*}}$ have the right weights $(1, 0)$ and $(0, 1)$, we can conclude that this construction leads exactly to a double Lie antialgebroid. To compare it with Mackenzie’s construction we need to apply Theorem 1.

Notice that $\Pi^2 T^*E$ coincides with $T^*\Pi E$. Hence the double vector bundle

$$\begin{array}{ccc} \Pi^2 T^*E = \Pi^2 T^*E^* & \longrightarrow & \Pi E^* \\ \downarrow & & \downarrow \\ \Pi E & \longrightarrow & M \end{array} \quad (70)$$

obtained by the complete parity reversion of the double vector bundle (68) is identical with (69). It remains to identify the respective homological vector fields on the total space, which is achieved by a direct inspection. We arrive at the following statement.

Proposition 6. *Roytenberg’s and Mackenzie’s pictures give the same notion of a double of a Lie bialgebroid.*

We can now identify the two constructions and speak simply of the (*cotangent*) *double* of a Lie bialgebroid. The Proposition shows that the cotangent double is fundamental and should be regarded as the correct extension of Drinfeld’s double to Lie bialgebroids.

Remark 6.1. The canonical symplectic structure on $T^*\Pi E$ preserved by the Hamiltonian homological vector fields X_{H_E} and $X_{H_{E^*}}$ corresponds to the invariant scalar product on Drinfeld’s double $\mathfrak{d}(\mathfrak{b}) = \mathfrak{b} \oplus \mathfrak{b}^*$ of a Lie bialgebra \mathfrak{b} .

6.2. Multiple Lie algebroids. We have already mentioned that the methods of this paper allow us to consider a natural concept of multiple Lie algebroids. (A part of the motivation for doing that comes from the theory of doubles.) We shall give an outline of this theory.

First we need a language for describing multiple vector bundles. Fix a natural number n . To define n -fold vector bundles, consider vector spaces $V_r, V_{r_1 r_2}, V_{r_1 r_2 r_3}, \dots$, of arbitrary dimensions $d_r, d_{r_1 r_2}, d_{r_1 r_2 r_3}$, etc., numbered by increasing sequences $r_1 < \dots < r_k$, where $0 < k \leq n$ and all r_i run from 1 to n .

Example 6.1. When $n = 1$, we have just one vector space $V = V_1$. When $n = 2$, we have V_1, V_2 and V_{12} . For $n = 3$, we have 7 spaces: $V_1, V_2, V_3, V_{12}, V_{13}, V_{23}$, and V_{123} . In general the number of spaces is $2^n - 1$.

For convenience of notation let us fix linear coordinates on each of the spaces, denoting them $v_{(r)}^{i_r}, v_{(r_1 r_2)}^{i_{r_1 r_2}}$, etc. (Each index such as i_r runs over its own set of values, of cardinality equal to the dimension of the respective space.)

Definition 3. An n -fold vector bundle E over a base M is a fiber bundle $E \rightarrow M$ with the standard fiber

$$\prod_r V_r \times \prod_{r_1 < r_2} V_{r_1 r_2} \times \dots \times V_{12\dots n}$$

where the transition functions have the form:

$$\begin{aligned} v_{(r)}^{i_r} &= v_{(r)}^{i'_r} T_{i'_r}^{i_r}, \\ v_{(r_1 r_2)}^{i_{r_1 r_2}} &= v_{(r_1 r_2)}^{i'_{r_1 r_2}} T_{i'_{r_1 r_2}}^{i_{r_1 r_2}} + v_{(r_1)}^{i'_{r_1}} v_{(r_2)}^{i'_{r_2}} T_{i'_{r_2} i'_{r_1}}^{i_{r_1 r_2}}, \\ &\dots\dots\dots \\ v_{(12\dots n)}^{i_{12\dots n}} &= v_{(12\dots n)}^{i'_{12\dots n}} T_{i'_{12\dots n}}^{i_{12\dots n}} + \dots + v_{(1)}^{i'_1} \dots v_{(n)}^{i'_n} T_{i'_n \dots i'_1}^{i_{12\dots n}}. \end{aligned}$$

In other words, the transformation for each of V_r is linear; for $V_{r_1 r_2}$ it is linear plus an extra term bilinear in V_{r_1} and V_{r_2} , etc.

Example 6.2. For a triple vector bundle $E \rightarrow M$ ($n = 3$), we have fiber coordinates: $v_{(1)}^{i_1}, v_{(2)}^{i_2}, v_{(3)}^{i_3}, v_{(12)}^{i_{12}}, v_{(13)}^{i_{13}}, v_{(23)}^{i_{23}}$, and $v_{(123)}^{i_{123}}$. The transformation law is as follows:

$$\begin{aligned} v_{(1)}^{i_1} &= v_{(1)}^{i'_1} T_{i'_1}^{i_1} \\ v_{(2)}^{i_2} &= v_{(2)}^{i'_2} T_{i'_2}^{i_2} \\ v_{(3)}^{i_3} &= v_{(3)}^{i'_3} T_{i'_3}^{i_3} \\ v_{(12)}^{i_{12}} &= v_{(12)}^{i'_{12}} T_{i'_{12}}^{i_{12}} + v_{(1)}^{i'_1} v_{(2)}^{i'_2} T_{i'_2 i'_1}^{i_{12}} \\ v_{(13)}^{i_{13}} &= v_{(13)}^{i'_{13}} T_{i'_{13}}^{i_{13}} + v_{(1)}^{i'_1} v_{(3)}^{i'_3} T_{i'_3 i'_1}^{i_{13}} \\ v_{(23)}^{i_{23}} &= v_{(23)}^{i'_{23}} T_{i'_{23}}^{i_{23}} + v_{(2)}^{i'_2} v_{(3)}^{i'_3} T_{i'_3 i'_2}^{i_{23}} \\ v_{(123)}^{i_{123}} &= v_{(123)}^{i'_{123}} T_{i'_{123}}^{i_{123}} + v_{(1)}^{i'_1} v_{(23)}^{i'_{23}} T_{i'_{23} i'_1}^{i_{123}} + v_{(2)}^{i'_2} v_{(13)}^{i'_{13}} T_{i'_{13} i'_2}^{i_{123}} + v_{(3)}^{i'_3} v_{(12)}^{i'_{12}} T_{i'_{12} i'_3}^{i_{123}} \\ &\quad + v_{(1)}^{i'_1} v_{(2)}^{i'_2} v_{(3)}^{i'_3} T_{i'_3 i'_2 i'_1}^{i_{123}} \end{aligned}$$

Remark 6.2. Triple vector bundles — with the quaternary case briefly mentioned — were introduced and studied in [14] from a different viewpoint (not using local trivializations and transition functions). Paper [14] also contains some ‘likely principles’ of duality for general multiple case. The existence of a local trivialization was not discussed in [14]; under a form of a “decomposition” it was explicitly introduced in the definition in [2]. To avoid any problems and since it is not our task to minimize axiomatic systems, here we define multiple vector bundles as a particular case of locally trivial fiber bundles from the start.

A multiple vector bundle has *faces*, which are also multiple vector bundles. A face is obtained by choosing indices $r_1 < \dots < r_k$; fiber coordinates for it will be the coordinates $v_{(r_1 \dots r_k)}^{i_{r_1 \dots r_k}}$ and all other coordinates with indices labelled by subsets of r_1, \dots, r_k . For example, for a triple vector bundle there are faces that are (ordinary) vector bundles and double vector bundles, corresponding to the edges and 2-faces of a 3-cube. In a natural way various partial projections and zero sections are defined.

The total space of a multiple vector bundle is a multi-graded manifold. More precisely, there are weights w_r , $r = 1, \dots, n$, each of them being a degree in all coordinates containing a given label r . For example, w_2 is the total degree in $v_{(2)}$, $v_{(12)}$, $v_{(23)}$, \dots , $v_{(12\dots n)}$. We define *total weight* as $w = w_1 + \dots + w_n$.

Due to the multilinearity of transition functions, for a multiple vector bundle the operations of *partial parity reversion* Π_r and *partial dual* D_r in the r -th direction, make sense for each $r = 1, \dots, n$.

Definition 4. An n -fold Lie antialgebroid E over a base M is an n -fold vector bundle $E \rightarrow M$ endowed with n odd vector fields Q_r of weights $(0, \dots, 1, \dots, 0)$ on the total space E such that

$$[Q_r, Q_s] = 0$$

for all r, s . (In particular, these fields are homological.)

Definition 5. An n -fold Lie algebroid E over a base M is an n -fold vector bundle $E \rightarrow M$ such that the n -fold vector bundle $\Pi^n E \rightarrow M$ obtained by the complete parity reversion $\Pi^n = \Pi_n \dots \Pi_1$ is an n -fold Lie antialgebroid.

In other words, we take the statement of Theorem 1 as a working definition for the multiple case.

Each face of a multiple Lie (anti)algebroid is also a multiple Lie (anti)algebroid.

We expect that it is possible to define multiple Lie algebroids also *à la* Mackenzie, via duals and bialgebroids, and to show the equivalence with Definition 5 (i.e., to prove the analog of Theorem 1). This will require an analysis of the structures induced on the neighbors of a multiple Lie (anti)algebroid.

6.3. More on doubles. Recall that Drinfeld's classical double of a Lie bialgebra is not just a Lie algebra, but also a coalgebra, and furthermore a Lie bialgebra again. This gives a direction in which to look in the case of Lie bialgebroids. Note that this second structure (for doubles of Lie bialgebroids) has not been discovered previously.

There is a conjectured statement that reads as follows.

General principle. Taking the double of an n -fold Lie bialgebroid gives an $(n + 1)$ -fold Lie bialgebroid, with additional properties such as a symplectic structure.

Of course it involves new notions yet to be defined. Multiple Lie algebroids were introduced above. As for “double Lie bialgebroids” (or “**bi**- double Lie algebroids”), and, further, the “**bi**-” multiple case, this is a subject of our forthcoming joint paper with Kirill Mackenzie. The example and discussion below should be seen just as preliminary hints.

Example 6.3. Consider again the double vector bundle given by (68). It is a double Lie algebroid. Notice that the core of it is the cotangent bundle $T^*M \rightarrow M$. Take the two duals of the double vector bundle (68). We obtain the double vector bundles

$$\begin{array}{ccc} TE & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \quad (71)$$

as the vertical dual and

$$\begin{array}{ccc} TE^* & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ TM & \longrightarrow & M \end{array} \quad (72)$$

as the horizontal dual. Both (71) and (72) are known to be double Lie algebroids as well [12]. The three double vector bundles in duality (68), (71), (72)

$$\begin{array}{ccccc} & & & TE^* & \\ & & & \downarrow & \\ TE & \longrightarrow & TM & \swarrow & \\ \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{T^*E} & M & \swarrow & E^* \end{array} \quad (73)$$

(we use the picture of a corner [14] for them), which are all double Lie algebroids, can be said to be making a ‘bi- double Lie algebroid’. One can mean by that, for example, that the triple vector bundle

$$\begin{array}{ccccc} & & T^*T^*E & \longrightarrow & TE^* \\ & & \downarrow & & \downarrow \\ TE & \xrightarrow{\quad} & TM & \swarrow & \\ \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{T^*E} & M & \swarrow & E^* \end{array} \quad (74)$$

is a triple Lie algebroid, as one can check.

To define a ‘bi- multiple Lie algebroid’ one can also use the supergeometry language for a shortcut. A **bi- n -fold Lie algebroid** (or an **n -fold Lie bialgebroid**) can be defined as an n -fold Lie algebroid E such that all its duals are also n -fold Lie algebroids satisfying the compatibility condition that reads as follows: on the total space with completely reversed parity $\Pi^n E$ there are n commuting homological vector fields Q_r of weights $w_r(Q_s) = \delta_{rs}$, which define the algebroid structures, and an odd or even (depending on the parity of the number n) Poisson bracket of weight $(-1, \dots, -1)$, and the fields Q_r are derivations of the bracket.

It should be possible to prove that this is equivalent to the cotangent double being an $(n+1)$ -fold Lie algebroid, as in Example 6.3. A precise relation of this multidimensional notion with Drinfeld’s theory is yet to be clarified.

Acknowledgement. I thank my friends Kirill Mackenzie, Yvette Kosmann-Schwarzbach and Hovhannes Khudaverdian, for inspiring discussions, most valuable criticism, and advice. K. Mackenzie has pioneered the whole subject of multiple and bi- structures in the groupoid and algebroid world. I thank most cordially Y. Kosmann-Schwarzbach for numerous comments and remarks that helped to improve the original version of the preprint [25] and were used in this text as well. My special thanks go to the organizers of the annual international Workshops on Geometric Methods in Physics in Białowieża, notably to A. Odziejewicz, for the highly inspiring atmosphere. Some of my early notes on the subject of this paper were made in Warsaw after the XXII Białowieża Workshop, and the main result was reported at the XXV Workshop. The earlier version of the present paper became known to experts as the preprint [25] and has already influenced some works. It was also reported at a special program at the Erwin Schroedinger

Institute in Vienna in September 2007 (see [26]). I thank the organizers of that program for their hospitality and the wonderful working atmosphere. I also wish to thank the anonymous referees for the careful reading of the text and important remarks.

REFERENCES

- [1] T. Courant. Tangent Lie algebroids. *J. Phys. A*, 27(13):4527–4536, 1994.
- [2] A. Gracia-Saz and K. C. H. Mackenzie. Duality functors for triple vector bundles. *Lett. Math. Phys.*, 90(1-3):175–200, 2009.
- [3] Y. Kosmann-Schwarzbach. Jacobian quasi-bialgebras and quasi-Poisson Lie groups. In *Mathematical aspects of classical field theory*, volume 132 of *Contemp. Math.*, pages 459–489. Amer. Math. Soc., 1992.
- [4] Y. Kosmann-Schwarzbach. Exact Gerstenhaber algebras and Lie bialgebroids. *Acta Appl. Math.*, 41:153–165, 1995.
- [5] Y. Kosmann-Schwarzbach. From Poisson algebras to Gerstenhaber algebras. *Ann. Inst. Fourier, Grenoble*, 46:1243–1274, 1996.
- [6] Zh.-J. Liu, A. Weinstein, and P. Xu. Manin triples for Lie bialgebroids. *J. Diff. Geom.*, 45:547–574, 1997.
- [7] K. C. H. Mackenzie. Double Lie algebroids and second-order geometry. I. *Adv. Math.*, 94(2):180–239, 1992. Available at: <http://www.kchmackenzie.staff.shef.ac.uk/dla1.pdf>.
- [8] K. C. H. Mackenzie. Double Lie algebroids and second-order geometry. II. *Adv. Math.*, 154(1):46–75, 2000, arXiv:dg-ga/9712013.
- [9] K. C. H. Mackenzie. Double Lie algebroids and the double of a Lie bialgebroid, arXiv:math.DG/9808081.
- [10] K. C. H. Mackenzie. Drinfel’d doubles and Ehresmann doubles for Lie algebroids and Lie bialgebroids. *Electron. Res. Announc. Amer. Math. Soc.*, 4:74–87 (electronic), 1998.
- [11] K. C. H. Mackenzie. On symplectic double groupoids and the duality of Poisson groupoids. *Internat. J. Math.*, 10(4):435–456, 1999.
- [12] K. C. H. Mackenzie. Notions of doubles for Lie bialgebroids. 2000, arXiv:math.DG/0011212.
- [13] K. C. H. Mackenzie. On certain canonical diffeomorphisms in symplectic and Poisson geometry. In *Quantization, Poisson brackets and beyond (Manchester, 2001)*, volume 315 of *Contemp. Math.*, pages 187–198. Amer. Math. Soc., Providence, RI, 2002.
- [14] K. C. H. Mackenzie. Duality and triple structures. In *The breadth of symplectic and Poisson geometry, Festschrift in Honor of Alan Weinstein*, volume 232 of *Progr. Math.*, pages 455–481. Birkhäuser Boston, Boston, MA, 2005, arXiv:math.SG/0406267.
- [15] K. C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.
- [16] K. C. H. Mackenzie. Ehresmann doubles and Drinfel’d doubles for Lie algebroids and Lie bialgebroids. 2006, arXiv:math.DG/0611799. 34 pages.
- [17] K. C. H. Mackenzie. Ehresmann doubles and Drinfel’d doubles for Lie algebroids and Lie bialgebroids *J. reine angew. Math.*, 658:193–245, DOI 10.1515/CRELLE.2011.092, 2011.
- [18] K. C. H. Mackenzie and P. Xu. Lie bialgebroids and Poisson groupoids. *Duke Math. J.*, 73(2):415–452, 1994.
- [19] K. C. H. Mackenzie and T. Mokri. Locally vacant double Lie groupoids and the integration of matched pairs of Lie algebroids. *Geom. Dedicata*, 77(3):317–330, 1999.
- [20] C. Roger. Algèbres de Lie graduées et quantification. In *Symplectic geometry and mathematical physics (Aix-en-Provence, 1990)*, volume 99 of *Progr. Math.*, pages 374–421. Birkhäuser Boston, Boston, MA, 1991.
- [21] D. Roytenberg. *Courant algebroids, derived brackets and even symplectic supermanifolds*. PhD thesis, UC Berkeley, 1999, arXiv:math.DG/9910078.
- [22] W. M. Tulczyjew. A symplectic formulation of particle dynamics. In *Differential Geometric Methods in Mathematical Physics, Bonn 1975*, volume 570 of *Lecture Notes in Math.*, pages 457–463. Springer-Verlag, Berlin, 1977.
- [23] A. Yu. Vaintrob. Lie algebroids and homological vector fields. *Uspekhi Matem. Nauk*, 52(2):428–429, 1997.
- [24] Th. Th. Voronov. Graded manifolds and Drinfeld doubles for Lie bialgebroids. In *Quantization, Poisson Brackets and Beyond*, volume 315 of *Contemp. Math.*, pages 131–168. Amer. Math. Soc., Providence, RI, 2002, arXiv:math.DG/0105237.

- [25] Th. Th. Voronov. Mackenzie theory and Q -manifolds. `arXiv:math/0608111v2 [math.DG]`.
- [26] Th. Th. Voronov. Q -manifolds and Mackenzie theory: an overview. ESI preprint 1952, 2007. `arXiv:0709.4232 [math.DG]`.
- [27] A. Weinstein. Symplectic groupoids and Poisson manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 16(1):101–104, 1987.
- [28] A. Weinstein. Coisotropic calculus and Poisson groupoids. *J. Math. Soc. Japan*, 40(4):705–727, 1988.

SCHOOL OF MATHEMATICS, UNIVERSITY OF MANCHESTER, OXFORD ROAD, MANCHESTER, M13 9PL,
UNITED KINGDOM

E-mail address: `theodore.voronov@manchester.ac.uk`